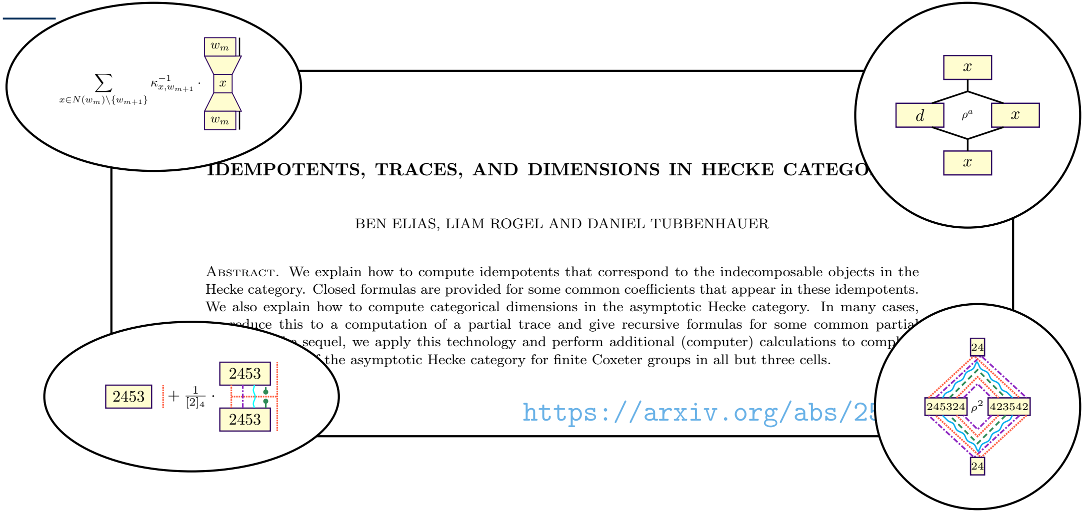
Idempotents and dimensions in the asymptotic Hecke category

Mini-conference: Categorification

Liam Rogel, RPTU

Norwich, January 13, 2026

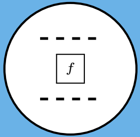
Motivation


IDEMPOTENTS, TRACES, AND DIMENSIONS IN HECKE CATEGO

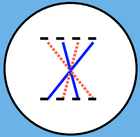
BEN ELIAS, LIAM ROGEL AND DANIEL TUBBENHAUER

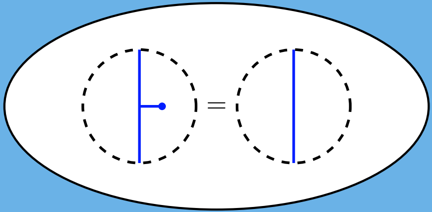
Abstract. We explain how to compute idempotents that correspond to the indecomposable objects in the Hecke category. Closed formulas are provided for some common coefficients that appear in these idempotents. We also explain how to compute categorical dimensions in the asymptotic Hecke category. In many cases, ...reduce this to a computation of a partial trace and give recursive formulas for some common partial ... sequel, we apply this technology and perform additional (computer) calculations to compl... ...f the asymptotic Hecke category for finite Coxeter groups in all but three cells.

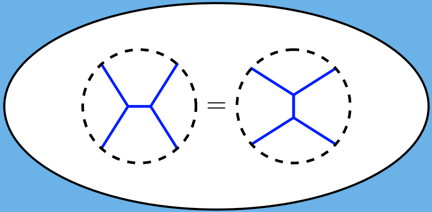
https://arxiv.org/abs/25



Diagrammatic Hecke Category







Soergel bimodules

Fix a Coxeter system (W, S).

Definition (Geometric representation)

- Let $V = \bigoplus_{s \in S} \mathbb{R}\alpha_s$ be with with bilinear form $(\alpha_s, \alpha_t) = -\cos(\pi / m_{s,t})$.
- Define the action $s.\lambda := \lambda - 2(\lambda, \alpha_s)\alpha_s$ and the *Demazure operator* $\partial_s(\lambda) := \frac{\lambda - s.\lambda}{\alpha_s}$.
- Consider $R := \mathrm{Sym}(V)$ as a polynomial ring $R = \mathbb{R}[\alpha_s \mid s \in S]$ with grading $\deg(\alpha_s) = 2$.

Category of Soergel bimodules

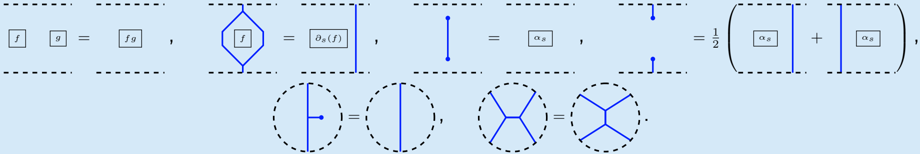
1. **Basic bimodules:** For each $s \in S$, define $B_s = R \otimes_{R^s} R(1)$ where R^s are the s-invariant polynomials.
2. **Bott–Samelson bimodules:** For $w = s_1 \cdots s_k$, define $BS(\underline{w}) = B_{s_1} \otimes_R \cdots \otimes_R B_{s_k}$.
3. **Soergel bimodules:** These are direct summands of finite direct sums of grading shifts of Bott–Samelson bimodules.
4. **Category $\mathbb{S}$Bim:** Objects are Soergel bimodules, morphisms are graded bimodule homomorphisms.

Diagrammatics

Definition (The *diagrammatic Category of Bott–Samelson bimodules* $\mathcal{H}_{BS}$)

- **Objects:** Sequences of colorful dots on a line. $s \mapsto$, $t \mapsto$, ...
- **Generating morphisms:** (R, R)-linearly generated by

- **Relations:** Multiplication, Keyhole, Barbell, Fusion, Frobenius unit/associativity:

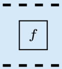


- And many more 2-color and 3-color relations.

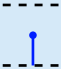
The diagrammatic Hecke category

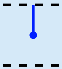
Theorem (The categories $\mathcal{H}_{BS}$ and $\mathbb{S}$Bim are equivalent)

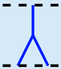
On objects:  $\mapsto B_s$ *and multiple dots to tensor products.*
On morphisms:

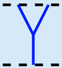


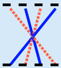
$\mapsto \big(R \to R(\deg(f)),\ 1 \mapsto f\big)\,;$


$\mapsto \big(B_s \to R(1),\ f \otimes g \mapsto fg\big)\,;$


$\mapsto \left(R \to B_s(1),\ 1 \mapsto \frac{1}{2}(1 \otimes \alpha_s + \alpha_s \otimes 1)\right)\,;$


$\mapsto \big(B_s \otimes B_s \to B_s(-1),\ 1 \otimes g \otimes 1 \mapsto \partial_s(g) \otimes 1\big)\,;$

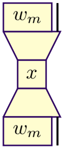

$\mapsto \big(B_s \to B_s \otimes B_s(-1),\ f \otimes g \mapsto f \otimes 1 \otimes g\big)\,;$

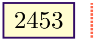

$\mapsto$ ***image of*** $B_{st\dots}$ ***in*** $BS(ts\dots)$.

The Karoubian envelope $\mathcal{H} := \mathrm{Kar}(\mathcal{H}_{BS})$ is the idempotent completion of $\mathcal{H}_{BS}$.

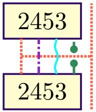
Computing idempotents

$$\sum_{x\in N(w_m)\setminus\{w_{m+1}\}} \kappa^{-1}_{x,w_{m+1}} \cdot$$





$$+ \frac{1}{[2]_4} \cdot$$



Direct sums

Definition

Let X be an object in an additive category. A *direct sum* $X \simeq M \oplus N$ is the data of morphisms

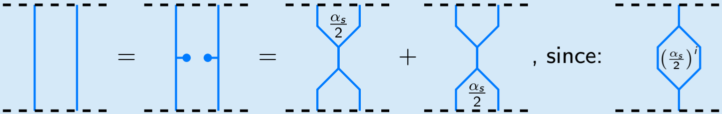
$$\iota_M \colon M \to X, \quad p_M \colon X \to M, \quad \iota_N \colon N \to X, \quad p_N \colon X \to N,$$

which satisfy

$$\mathrm{id}_X = \iota_M p_M + \iota_N p_N, \quad p_M \iota_N = 0, \quad p_N \iota_M = 0, \quad \mathrm{id}_N = p_N \iota_N, \quad \mathrm{id}_M = p_M \iota_M.$$

Example

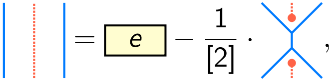
We have $BS(ss) \simeq B_s(-1) \oplus B_s(+1)$ via:



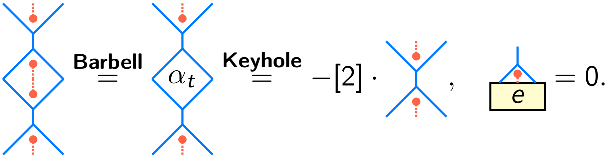
$$= \begin{cases} 0, & \text{if } i \in \{0,2\}, \\ \mathrm{id}_{B_s}, & \text{if } i = 1. \end{cases}$$

Dihedral idempotents

Let s, t be reflection with $m_{s,t} > 2$, set $[2] = -\partial_s(\alpha_t) = \partial_t(\alpha_s)$.
Then $B_s \otimes B_t \otimes B_s \simeq B_{sts} \oplus B_s$ can be seen diagrammatically as



where $\boxed{e}$ is the orthogonal idempotent projection to B_{sts} since:



(1)

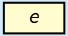
Clasp idempotents

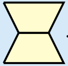
Definition

We call $X \in \mathcal{H}$ a *w-object* if it is self-dual and $X \simeq B_w \oplus \bigoplus_{y<w} B_y^{\oplus n_y}$ for $n_y \in \mathbb{N}[v^{\pm 1}]$.

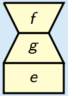
An element $e \in \mathrm{End}^0(X)$ is a *clasp idempotent* if:

1. **Identity coefficient:**



$= \mathrm{id}_X + \sum$

.

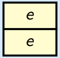
2. **Annihilation property:** For $\deg(f) + \deg(g) = 0$



$= 0.$

Theorem

If a clasp idempotent $e \in End^0(X)$ exists, then it is unique and satisfies:

Idempotency:


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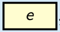
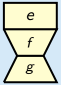
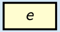
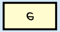


, *Pre-annihilation:*


$= 0$, *flip-invariance:*


$=$


Idempotent computation

Construction

Let $\underline{w} = (i_1, \ldots, i_n)$ be a reduced expression for w and assume that an idempotent e_n for w has been constructed. Let i_{n+1} be such that (w, i_{n+1}) is also reduced, we compute an idempotent e_{n+1}:

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1. Compute the tensor product:

$$B_w \otimes B_{i_{n+1}} \simeq \bigoplus_{x \in N(w)} B_x^{k_x}.$$

   We assume here $k_x = 1$.

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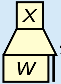
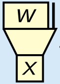
2. For each $x \in N(w)$, find projection and inclusion:

$p^x_{w,i_{n+1}} =$ 

, $\quad \iota^x_{w,i_{n+1}} =$ 

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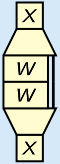
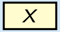
3. For each x compute the scalars $\kappa_{x,n+1}$ such that:

$p^x_{w,i_{n+1}} \circ \iota^x_{w,i_{n+1}} =$ 

$= \kappa_{x,n+1} \cdot$ 

Note that $\kappa_{(w,i_{n+1}),n+1} = 1$.

RPTU

Idempotent computation

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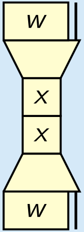
4. Define e_{n+1} recursively as:

$$\boxed{e_{n+1}} := \boxed{e_n}\,\Big| - \sum_{x \in N(w) \setminus \{(w, i_{n+1})\}} \kappa_{x,n+1}^{-1} \cdot$$

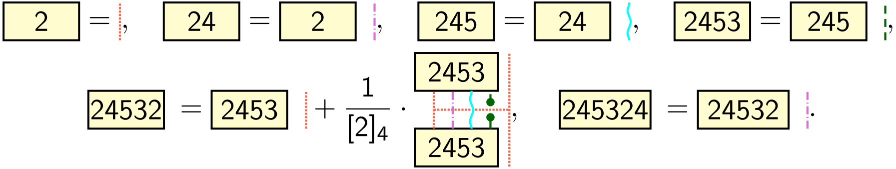


Idempotents in type F_n

As a short hand, we write i instead of s_i. Consider the element $\underline{x} = (2,4,5,3,2,4)$. In the branching graph we visualize the decomposition of $B_{x_m} \otimes B_{i_{m+1}}$:

$$\Gamma(243524) = \emptyset \to 2 \to 24 \to 245 \rightleftarrows 2453 \to 24532 \to 245324$$

The idempotens have the form:



Idempotents in type H_3

Let $\underline{d} = 232$ and $\underline{x} = \underline{d}123$. The branching graph is:

$$\Gamma(\underline{d}123) = \ \emptyset \rightarrow 2 \rightleftarrows 23 \rightarrow \underline{d} \rightleftarrows \underline{d}1 \rightleftarrows \underline{d}12 \rightarrow \underline{d}123$$

Idempotents in type H_3

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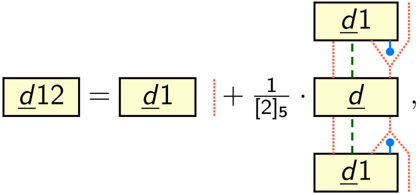

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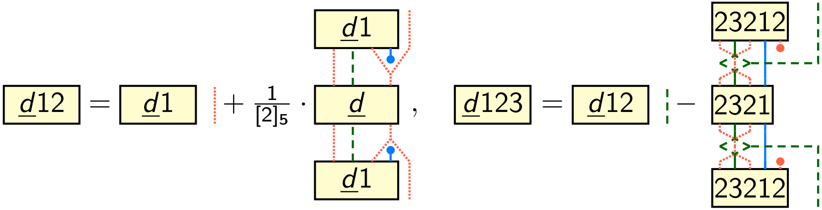

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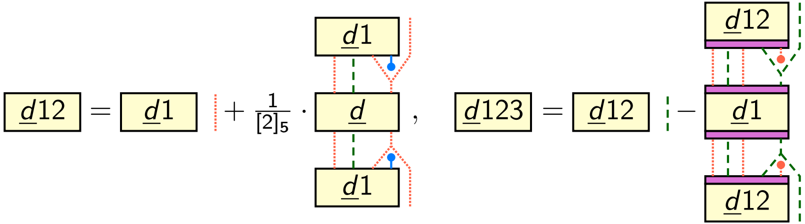
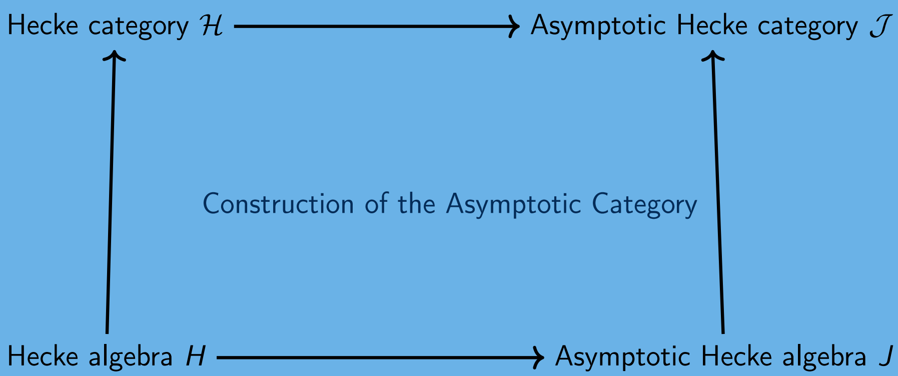
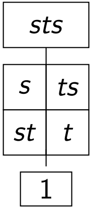




Hecke category $\mathcal{H}$
Asymptotic Hecke category $\mathcal{J}$
Construction of the Asymptotic Category
Hecke algebra H
Asymptotic Hecke algebra J

Cells and the Asymptotic Algebra

For a Coxeter system (W, S) let $H(W)$ be the Hecke algebra (over $\mathbb{Z}[v^{\pm 1}]$) with Kazhdan–Lusztig basis $\{b_w \mid w \in W\}$. The decomposition of the product of the b_w's partitions the group W into cells. E.g. type $I_2(3)$:
Big blocks are called *J-cells*, small blocks *H-cells*. The J-cells are $\{1\}$, $\{s, t, st, ts\}$, and $\{sts\}$. We have for example:



$$b_s b_s = (v + v^{-1}) b_s, \qquad b_s b_t = b_{st}, \qquad b_{st} b_s = b_{sts} + b_s$$

In the asymptotic algebra:

$$j_s j_s = j_s, \qquad j_s j_t = 0, \qquad j_{st} j_{ts} = j_s$$

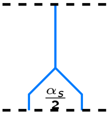
The *asymptotic Hecke algebra* is generated by j_x for $x \in W$, where we kill all summands not from the same cell and lowest degree.

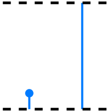
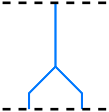
Motivation for the asymptotic Hecke category

To categorify the asymptotic Hecke algebra J, we want to change $\mathcal{H}$ into a new category $\mathcal{J}$ such that

$$B_s \otimes B_s \simeq B_s(+1) \oplus B_s(-1) \quad \text{becomes} \quad J_s \otimes J_s \simeq J_s.$$

Problem: The direct sum is *not* canonical. We can modify p_{+1} and i_{-1} in the direct sum decomposition:

$p_{+1} =$ 

, $p'_{+1} =$  $+ \lambda\alpha_s$  .

To imitate the construction from the algebra level, we need to relate different graded summands.

Relative Hard Lefschetz for Soergel bimodules

For $B \in \mathbb{S}\mathrm{Bim}$, let $\tau_{\leq i} B$ denote the summands of degree $\leq i$. We can project onto them canonically. The *perverse cohomology* is defined as $H^i(B) = (\tau_{\leq i} B / \tau_{\leq i-1} B)(-i)$.

Theorem (Relative Hard Lefschetz, Elias–Williamson '16)

Let $x, y \in W$ be arbitrary and let ρ be dominant regular ($\partial_s(\rho) = 1$ for all s). The multiplication map

$$\eta : B_x \otimes_R B_y \to B_x \otimes_R B_y(2), \quad b \otimes b' \mapsto b \otimes \rho b' = b\rho \otimes b'$$

induces isomorphisms (for all $i \geq 0$)

$$\eta^i : H^{-i}(B_x \otimes_R B_y) \xrightarrow{\sim} H^i(B_x \otimes_R B_y).$$

For any j_z summand of $j_x j_y$ in a-value k, we can use the Lefschetz operator with the canonical projection $B_x \otimes B_y \to B_z(-k)$ and inclusion $B_z(+k) \to B_x \otimes B_y$ to define a new monoidal category $\mathcal{J}$.

Construction of $\mathcal{J}_c$

Let W be a Coxeter group, $c \subseteq W$ a two-sided cell, and a the value of the a-function on c. Assume ρ is dominant regular. We construct the asymptotic category $\mathcal{J}_c$ as follows:

1. $\mathcal{H}_{\geq c}$ is the category with objects $B_z(m)$ for $z \geq c$ and $m \in \mathbb{Z}$ (no monoidal product).
2. $I_{>c}$ is the tensor ideal of all morphisms factoring over objects from bigger cells.
3. $\mathcal{H}'_c$ is the quotient $\mathcal{H}_c / I_{>c}$.
4. $\mathcal{J}_c$ is the subcategory of $\mathcal{H}'_c$ generated by $J_x := B_x$ for $x \in c$ with monoidal product

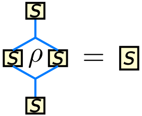
$$B \star B' := H^{-a}(B \otimes B').$$

For h a diagonal H-cell we also write $\mathcal{J}_h$. This is a fusion category.

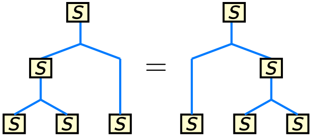
A trivial asymptotic category

Consider type A_1, so the group $W = \langle s \mid s^2 = 1 \rangle$. Then $c = h = \{s\}$ is a J/H-cell.

We have $J_s \otimes J_s \simeq J_s$:



The associator is trivial:



This is a fusion category with only one object. Hence the asymptotic category is $\mathcal{J}_c \simeq \mathrm{Vec}$.

A non-trivial asymptotic category

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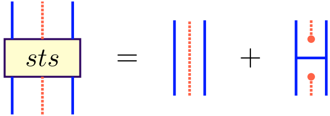

$\in \mathcal{J}_c.$

RPTU

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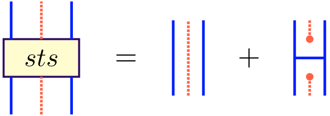
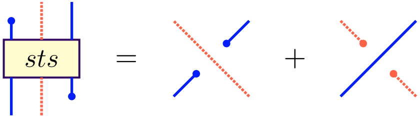
$0 =$ 

$\in \mathcal{J}_c$.

Specifically, composing with startdot and enddot on opposite sides yields:

$0 =$ 

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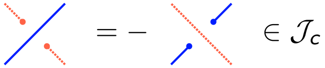

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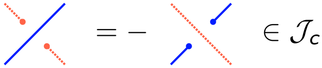
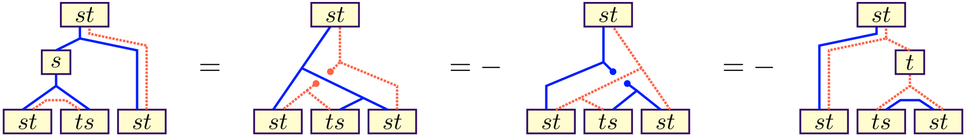
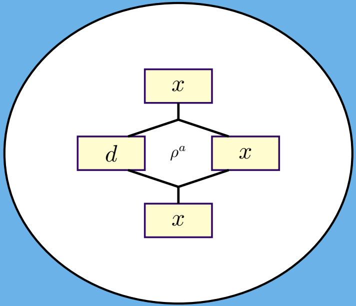
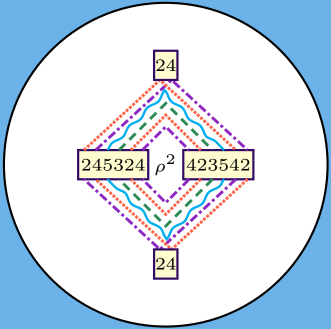


We can use this relation to show that there is a non-trivial $6j$-symbol:



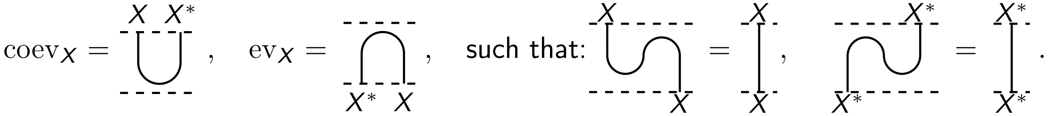
Dimensions in the asymptotic Category





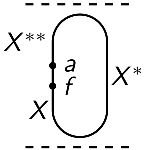
Dimension of objects

Let $\mathcal{C}$ be a $\Bbbk$-linear fusion category with *pivotal structure*, i.e. a collection of isomorphisms $a_X : X \xrightarrow{\sim} X^{**}$ natural in X with $a_{X \otimes Y} = a_X \otimes a_Y$. For $X \in \mathcal{C}$ a dual is an object X^* if there are maps $\mathrm{coev}_X : \Bbbk \to X \otimes X^*$ and $\mathrm{ev}_X : X^* \otimes X \to \Bbbk$ written



For a morphism $f : X \to X$ we define the trace as $\mathrm{tr}(f) \in \Bbbk$ and the dimension as $\dim(X) = \mathrm{trace}(\mathrm{id}_X)$:

$\mathrm{tr}(f)\,\mathrm{id}_1 =$

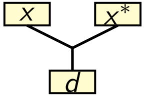


Dimensions in the asymptotic Category

Let $h = \{x_0 = d, x_1, \ldots, x_n\}$ be an H-cell of a-value k. We want to compute the dimensions of the objects in $\mathcal{J}_h$.

Dimensions in the asymptotic Category

Let $h = \{x_0 = d, x_1, \ldots, x_n\}$ be an H-cell of a-value k. We want to compute the dimensions of the objects in $\mathcal{J}_h$. For $x \in h$ we denote a degree $-k$ map including B_d into $B_x \otimes B_{x^*}$ by

$\mathrm{incl}_x =$ 

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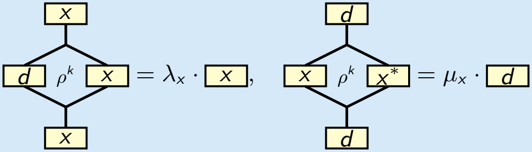
Dimensions in the asymptotic Category

Let $h = \{x_0 = d, x_1, \dots, x_n\}$ be an H-cell of a-value k. We want to compute the dimensions of the objects in $\mathcal{J}_h$. For $x \in h$ we denote a degree $-k$ map including B_d into $B_x \otimes B_{x^*}$ by incl_x.

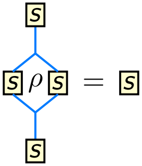
Theorem

Fix a dominant regular element ρ and $x \in h$. Fix $\mathrm{incl}_x \colon B_d \to B_x \otimes B_{x^*}$ of degree $-k$. Then λ_x and μ_x (defined by the diagrams below) are nonzero if and only if $\mathrm{incl}_x \notin I_c$, in which case the ratio μ_x/λ_x is independent of ρ and equals the categorical dimension of J_x in $\mathcal{J}_c$.



Example: Type A_1

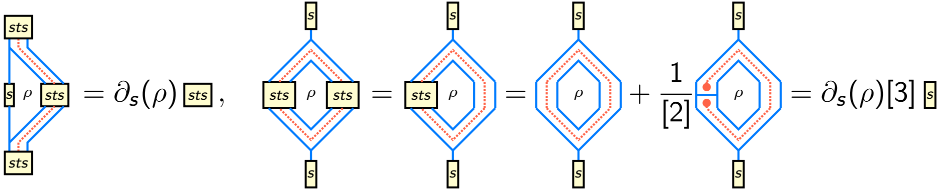
Consider the cell $h = \{s\}$ of a-value 1 in type A_1.
We choose $\rho = \frac{\alpha_s}{2}$ and compute:



Therefore, the dimension of J_s in $\mathcal{J}_h$ is 1.

Example: Type $I_2(5)$

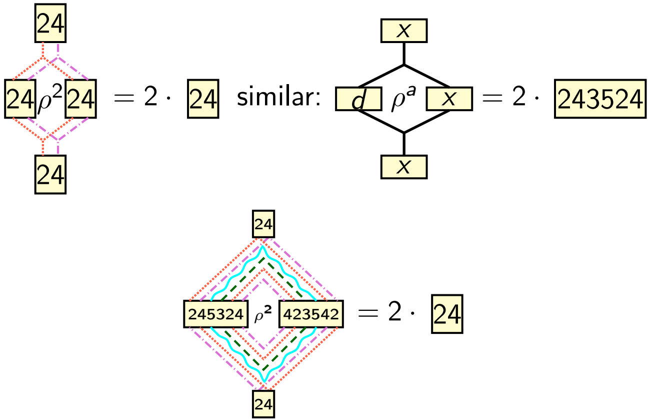
Consider the cell $h = \{s, sts\}$ of a-value 1 in type $I_2(5)$. We have $j_{sts}^2 = j_s + j_{sts}$. We choose $\rho = \frac{\alpha_s}{2} + \frac{\alpha_t}{2}$ and compute:



Therefore, the dimension of J_{sts} in $\mathcal{J}_h$ is $[3] = \frac{1+\sqrt{5}}{2}$.

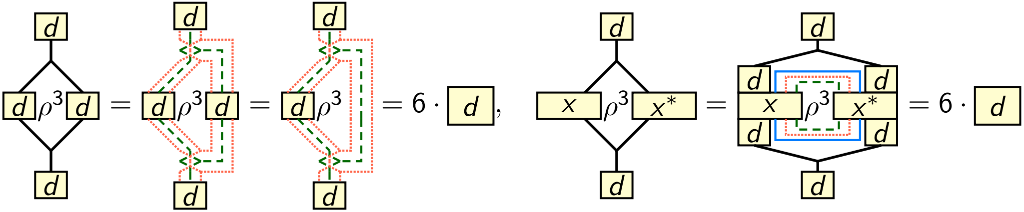
Example: Type F_n

Consider the cell $h = \{d = 24, x = 243524\}$ of a-value 2 in type F_n for $n \geq 5$. The dimension of A_x is again 1:



Example: Type H_3

Consider the cell $h = \{d = 232, x = 232123\}$ of a-value 3 in type H_3. We can choose $\rho = \frac{9\varphi+6}{2}\alpha_1 + (5\varphi+4)\alpha_2 + \frac{5\varphi+5}{2}\alpha_3$ and the dimension of J_x is also 1:



One step in the example of type H_3