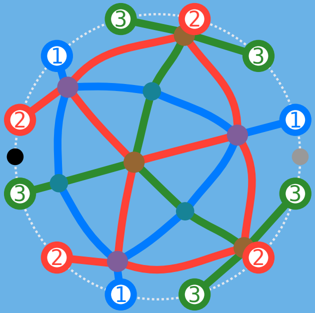
Circular Leaves for the diagrammatic Hecke category in type A_3

Tensor categories in representation theory & vice versa, Uppsala University



Liam Rogel, RPTU University

26 August 2026

Diagrammatic Hecke Category

Fix a Coxeter system (W, S) and set $R \coloneqq \mathbb{R}[\alpha_i \mid i \in S]$, with $i.\alpha_j \coloneqq \alpha_j - 2\cos(\pi/m_{ij})\,\alpha_i$ and $\partial_i(f) \coloneqq \frac{f - i.f}{\alpha_i}$.

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> **Objects:** words in S as coloured marks on a line: 121321 $\mapsto$ -1- -2- -1- -3- -2- -1-

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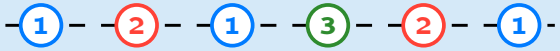
- **Objects:** words in S as coloured marks on a line: $121321 \mapsto$ 
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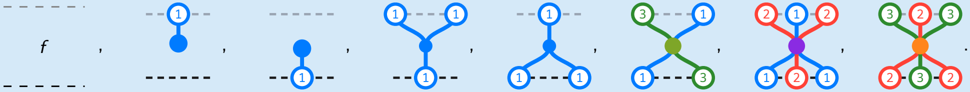

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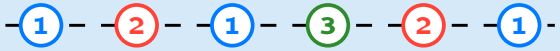
> **Objects:** words in S as coloured marks on a line: $121321 \mapsto$ 

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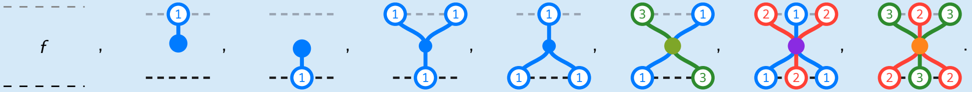

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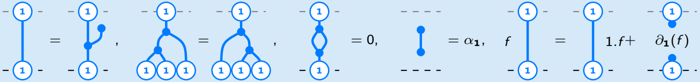

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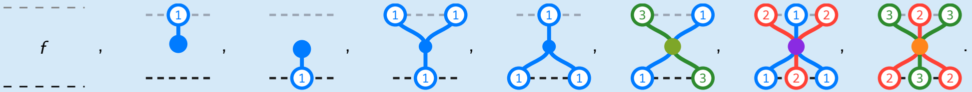

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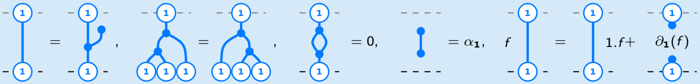

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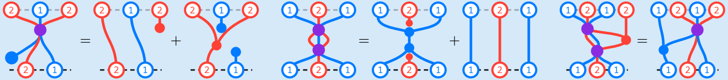

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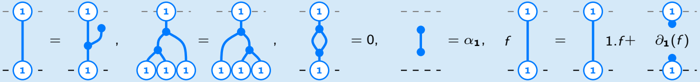

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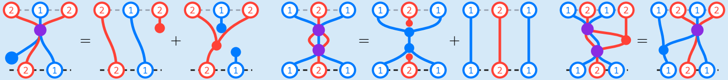
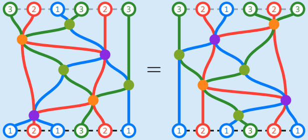


> **One three-colour relation, the Zamolodchikov in A_3:**



A basis: double leaves

Theorem (Libedinsky 2008; Elias–Williamson 2016)

The Hom space $\mathrm{Hom}_{\mathcal{H}_{BS}}(x, y)$ between words x and y has, as a left R-module, a basis given by the double leaves $\mathbb{DL}(x, e \mid y, f)$, indexed by the subexpressions e, f with $x^e = y^f$. There is an inductive algorithm: at step i it depends on whether the expression $x_{\mathbf{1}..i+\mathbf{1}}^{e_{\mathbf{1}..i+\mathbf{1}}}$ is *longer* or *shorter* than $x_{\mathbf{1}..i}^{e_{\mathbf{1}..i}}$, and on the value of $e_{i+\mathbf{1}}$.

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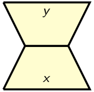

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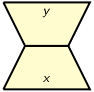

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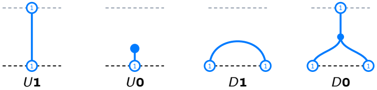
The four steps of $\mathrm{LL}(x, e)$: 

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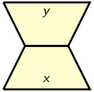

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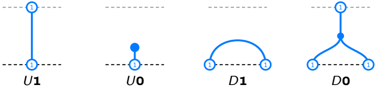

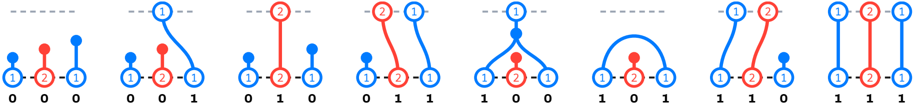

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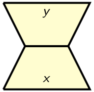

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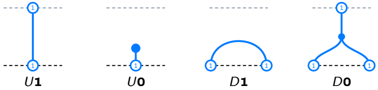

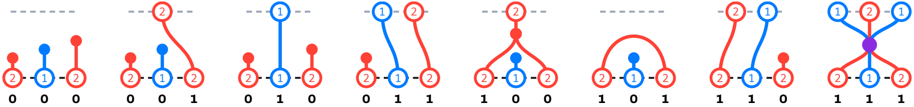

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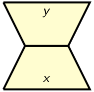

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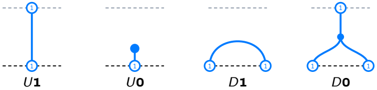

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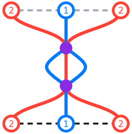

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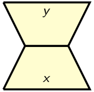

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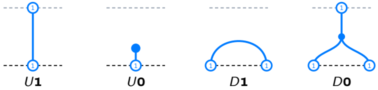
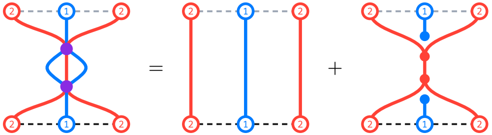
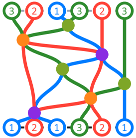
The four steps of $\mathrm{LL}(x, e)$: 

$\mathbb{DL}(212, 111 \mid 212, 111) =$ 

Diagrammatic morphisms are planar graphs



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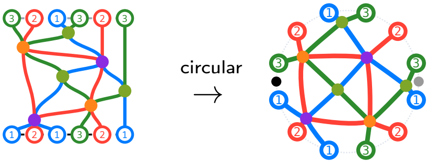

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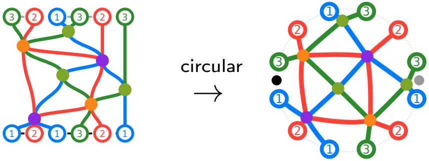

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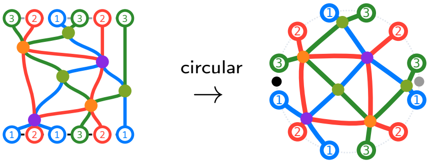

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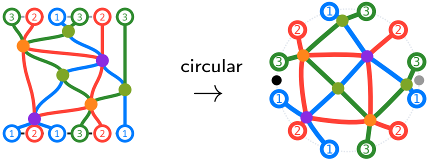

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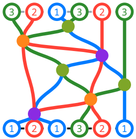
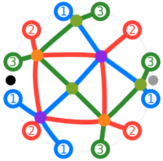


circular
$\longrightarrow$



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1-dot, 2-dot, 3-dot, 1-triv., 2-triv., 3-triv., 12-braid, 13-braid, 23-braid.

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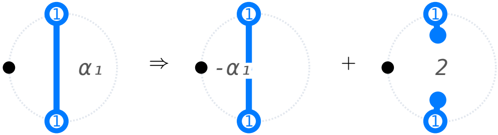

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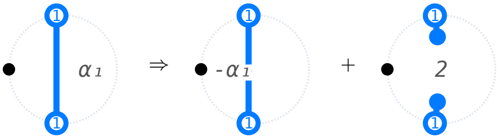

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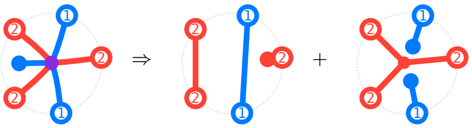

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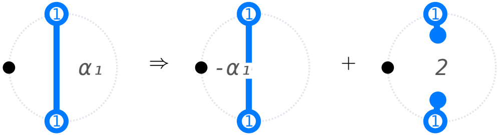

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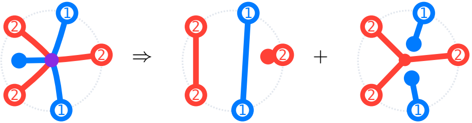
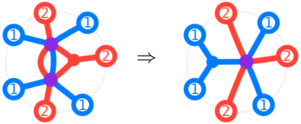


two-colour associativity



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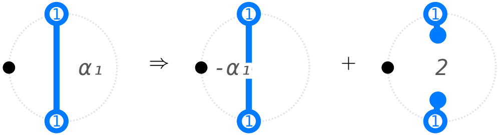

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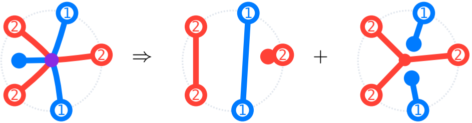

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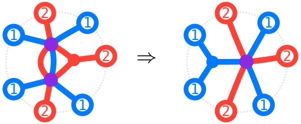

Question. Just keep simplifying: do we land in a normal form?

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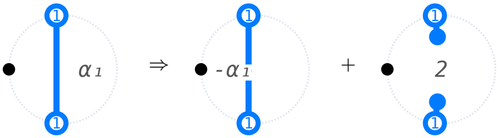
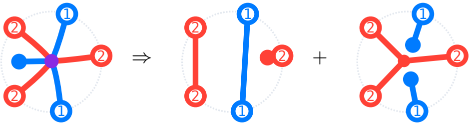
Do we get a normal form?

Idea. Give every relation a direction: $= \rightsquigarrow \Rightarrow$. Use naive moves to make the diagram simpler.

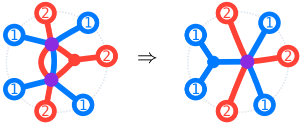
fusion: the polynomial moves *towards* the marking



dots: we pull dots through



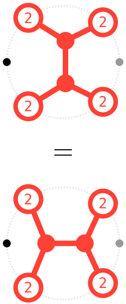
two-colour associativity



Question. Just keep simplifying: do we land in a normal form? **Answer. NO, STILL VERY FAR AWAY.**

Problems and solutions

Problem 1: Frobenius associativity on 2222.

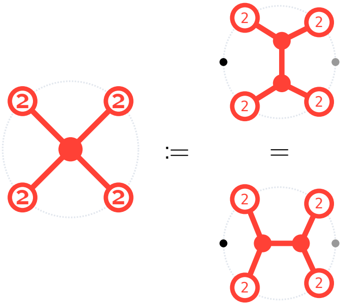


RPTU

Problems and solutions

Problem 1: Frobenius associativity on 2222.

We introduce a general 2-node with an arbitrary number of 2-arms (≥ 3).

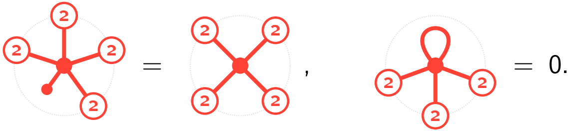


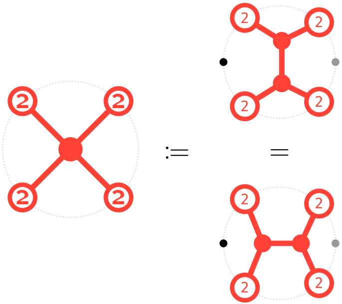
Problems and solutions

Problem 1: Frobenius associativity on 2222.

We introduce a general 2-node with an arbitrary number of 2-arms (≥ 3).

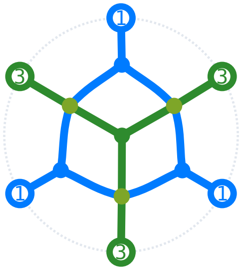
One needs to define generalizations of the rules, i.e. a general dot and needle relation:





Problems and solutions

Problem 2: We may not see needles.



two-colour associativity on 13 is all we have:

Problems and solutions

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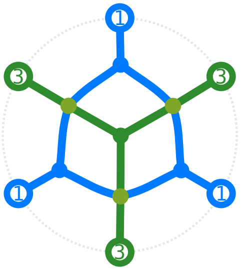

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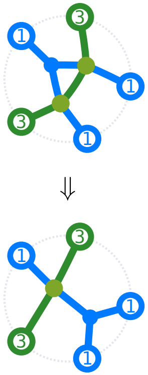

RPTU

Problems and solutions

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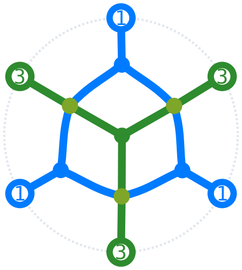
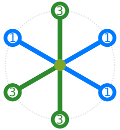


We introduce the general 13-node with an arbitrary number of arms of each colour (at least 3 in total, at least 2 per colour).



As an exercise for the bored listener: write down the dot and merge rules for the general 13-node.

Problems and solutions

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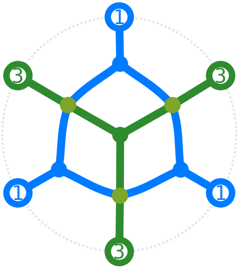

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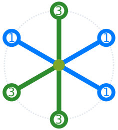

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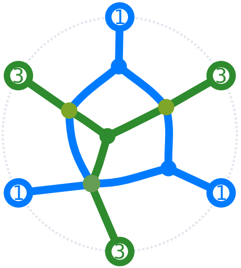

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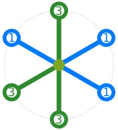

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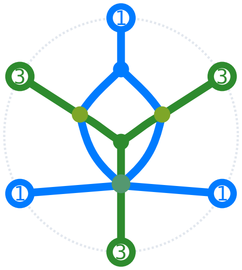

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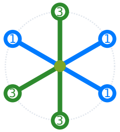

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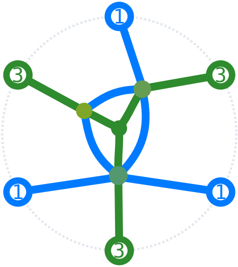

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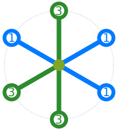

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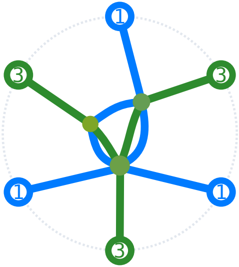

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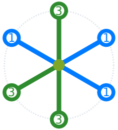

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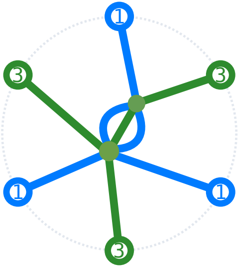

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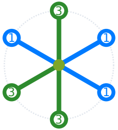

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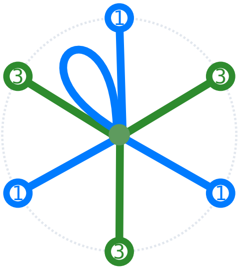

$$= 0$$

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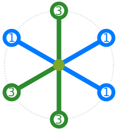

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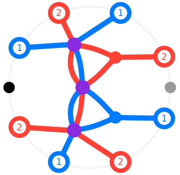
RPTU

Problems and solutions

Problem 3: The order of the moves matter.



RPTU

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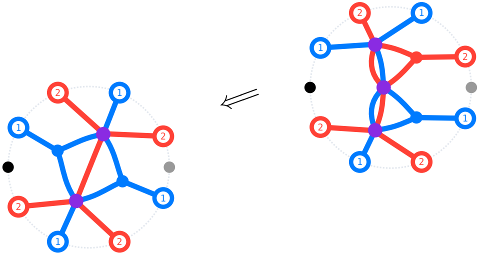

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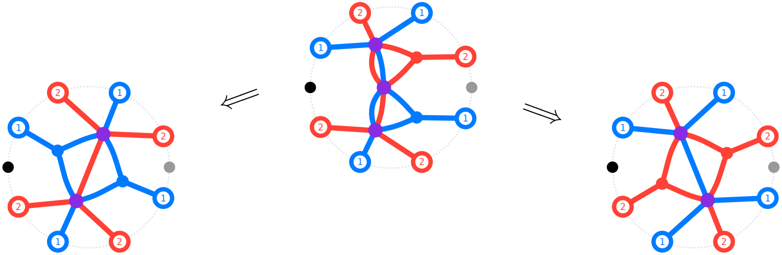

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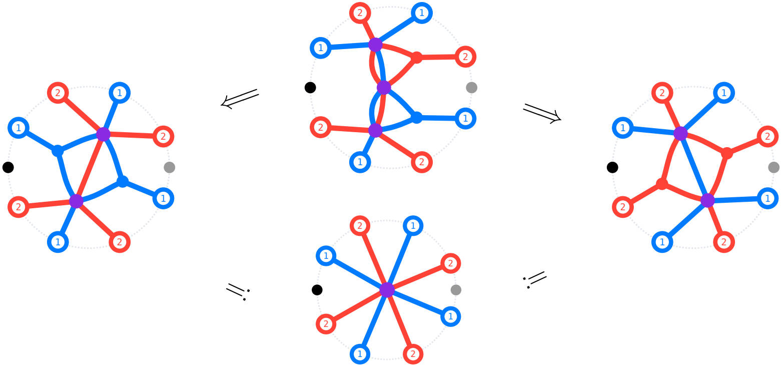
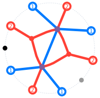


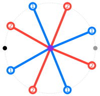
RPTU

The general 12-node

We define the general 12-node (and similarly the 23-node) with $2k$ arms as follows.



$=:$



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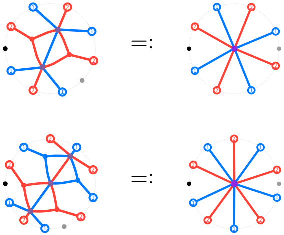

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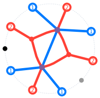
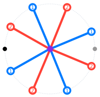
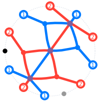
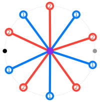
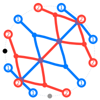
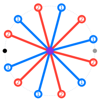
 $=:$ 

 $=:$ 

 $=:$ 

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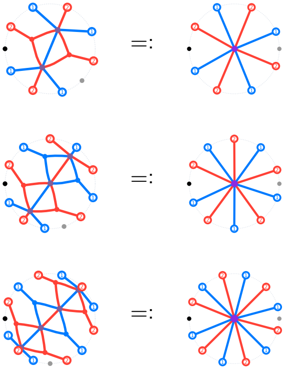
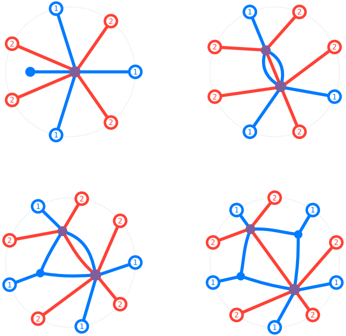


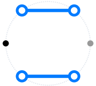
Exercise: Write down how the following diagrams simplify.



Parallel edges and 3-dot relation

1. **Parallel edges are not always double leaves.**

   We have a double leaf

$$\mathbb{DL}(11, 11 \mid 11, 11) =$$  $\mathrm{id}_{\mathbf{11}} =$

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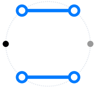
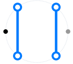
$\mathbb{DL}(11, 11 \mid 11, 11) =$  $\quad \mathrm{id}_{\mathbf{11}} =$ 

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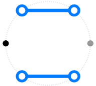
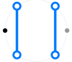
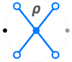
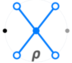
$\mathbb{DL}(11, 11 \mid 11, 11) =$ 

$\mathrm{id}_{\mathbf{11}} =$  $=$ 

$+$ 

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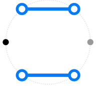
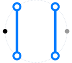
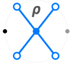
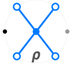
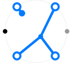
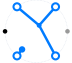
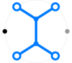
$\mathbb{DL}(11, 11 \mid 11, 11) =$ 

$\mathrm{id}_{\mathbf{11}} =$  $=$  $+$ 

$=$  $+$  $- \alpha_{\mathbf{1}} \cdot$ 

RPTU

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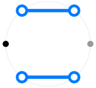
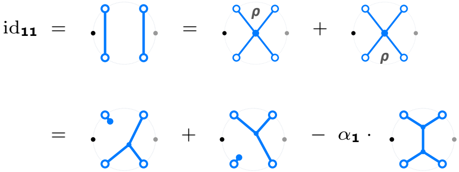
$\mathbb{DL}(11, 11 \mid 11, 11) =$ 



$$= \mathbb{DL}(11, 10 \mid 11, 01) + \mathbb{DL}(11, 01 \mid 11, 10) - \alpha_{\mathbf{1}}\, \mathbb{DL}(11, 10 \mid 11, 10).$$

RPTU

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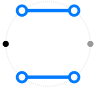
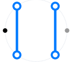
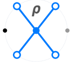
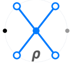
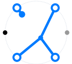
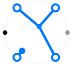
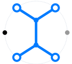
$\mathbb{DL}(11, 11 \mid 11, 11) =$ 

$\mathrm{id}_{\mathbf{11}} =$  $=$ 

$+$ 

$=$  $+$  $- \alpha_{\mathbf{1}} \cdot$ 

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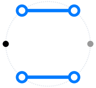
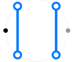
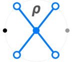
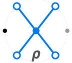
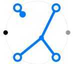
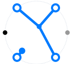
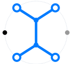
$\mathbb{DL}(11, 11 \mid 11, 11) =$ 

$\mathrm{id}_{\mathbf{11}} =$  $=$ 

$+$ 

$=$  $+$  $- \alpha_{\mathbf{1}} \cdot$ 

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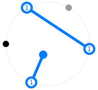
$\mathbb{DL}(11, 01 \mid 1, 1) =$ 

RPTU

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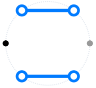

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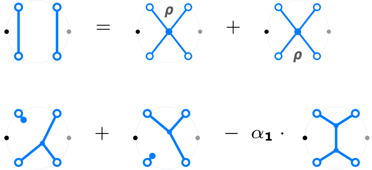

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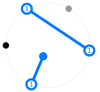

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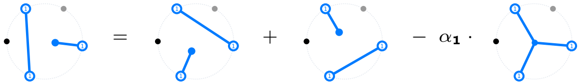

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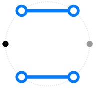
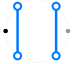
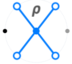
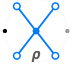
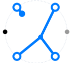
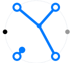
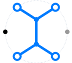
$\mathbb{DL}(11,11 \mid 11,11) =$ 

$\mathrm{id}_{\mathbf{11}} =$  $=$ 

$+$ 

$=$  $+$  $- \alpha_{\mathbf{1}} \cdot$ 

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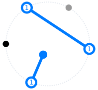
$\mathbb{DL}(11,01 \mid 1,1) =$ 

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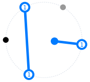
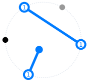
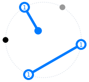
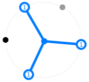
 $=$  $+$  $- \alpha_{\mathbf{1}} \cdot$ 

This is what we call the 3-dot relation. And we also have a general condition on what dots can see, and how to simplify.

RPTU

Circular leaves

Definition

A circular leaf for a word w is a planar diagram with leaf word w, consisting of general 2-, 12-, 13- and 23-nodes, which cannot be simplified any further, using the rules described before, and on which the general 3-dot condition, the general parallel condition and the general Zamolodchikov condition hold.

RPTU

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6. there are no Zamolodchikov subgraphs.

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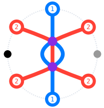

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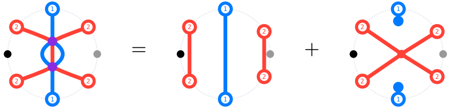

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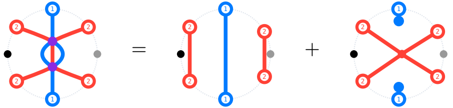

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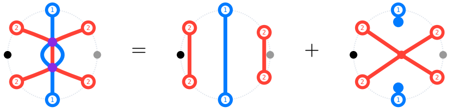
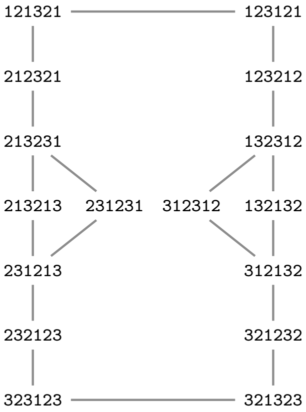
	$\mathbb{CL}(111111)$	$\mathbb{CL}(100001)$
$\mathbb{DL}(212, 111 \mid 212, 111)$	1	1
$\mathbb{DL}(212, 100 \mid 212, 100)$	0	1



RPTU

Zamolodchikov

The longest word in type A_3, $w_0 = 121321$, has 16 different reduced expressions.



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$$
\begin{tikzcd}
121321 \arrow[rr, dash] \arrow[d, dash] & & 123121 \arrow[d, dash] \\
212321 \arrow[d, dash] & & 123212 \arrow[d, dash] \\
213231 \arrow[d, dash] & & 132312 \arrow[d, dash] \\
213213 \arrow[d, dash] & & 132132 \arrow[d, dash] \\
231213 \arrow[d, dash] & & 312132 \arrow[d, dash] \\
232123 \arrow[d, dash] & & 321232 \arrow[d, dash] \\
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\texttt{6:232123} \arrow[d, dash] & \texttt{9:321232} \arrow[d, dash] \\
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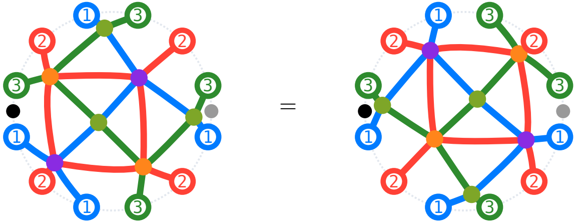
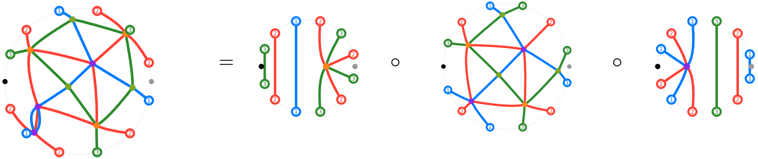
6:232123	9:321232
7:323123	8:321323

The ordinary Zamolodchikov relation $\mathrm{Zamo}(1,8) = \overline{\mathrm{Zamo}(8,1)}$:



The $\mathrm{Zamo}(2, 9)$ variant

$\mathrm{Zamo}(8, 9) \circ \mathrm{Zamo}(1, 8) \circ \overline{\mathrm{Zamo}(1, 2)}$:



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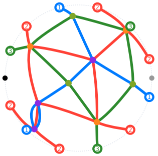

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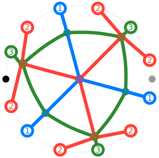

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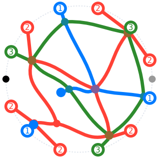

Elias–Jones–Wenzl

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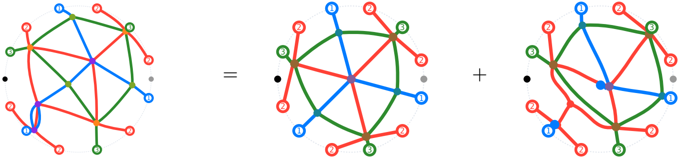
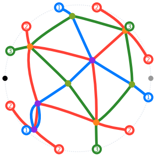


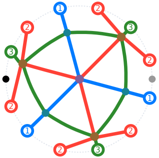
13-dot

The $\mathrm{Zamo}(2,9)$ variant

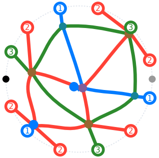
$\mathrm{Zamo}(8,9) \circ \mathrm{Zamo}(1,8) \circ \overline{\mathrm{Zamo}(1,2)}$:



$=$



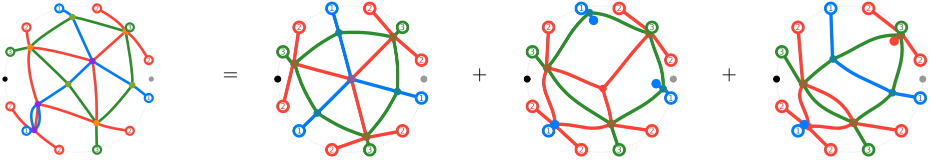
$+$



2-merge

The $\mathrm{Zamo}(2,9)$ variant

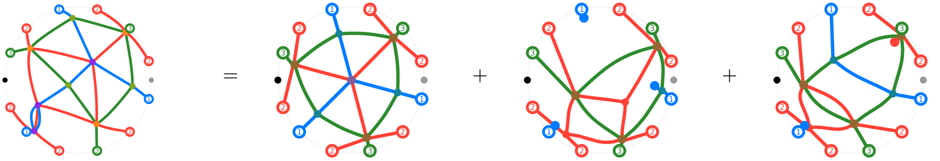
$\mathrm{Zamo}(8,9) \circ \mathrm{Zamo}(1,8) \circ \overline{\mathrm{Zamo}(1,2)}$:



12-dot into braid

The $\mathrm{Zamo}(2, 9)$ variant

$\mathrm{Zamo}(8, 9) \circ \mathrm{Zamo}(1, 8) \circ \overline{\mathrm{Zamo}(1, 2)}$:

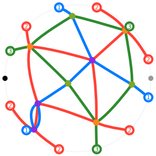


13-dot

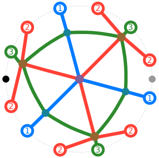
RPTU

The $\mathrm{Zamo}(2,9)$ variant

$\mathrm{Zamo}(8,9) \circ \mathrm{Zamo}(1,8) \circ \overline{\mathrm{Zamo}(1,2)}$:



$=$



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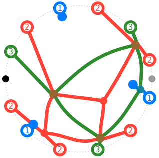
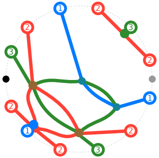
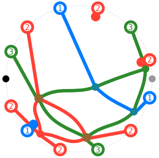


$+$



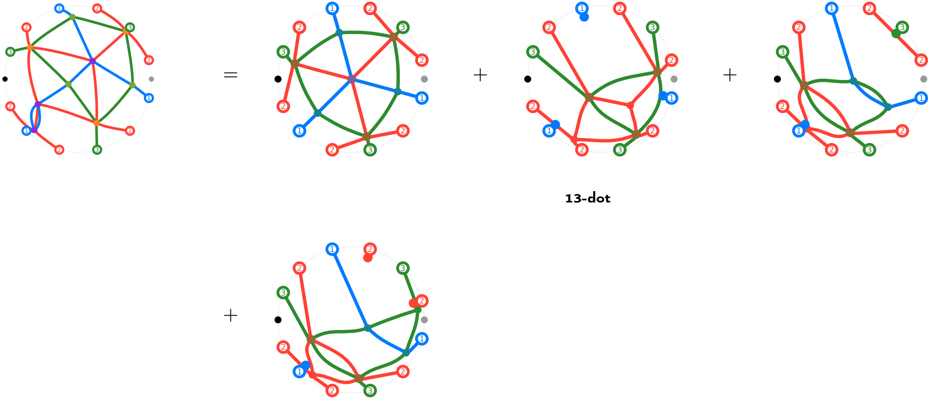
23-dot into braid

$+$



The $\mathrm{Zamo}(2,9)$ variant

$\mathrm{Zamo}(8,9) \circ \mathrm{Zamo}(1,8) \circ \overline{\mathrm{Zamo}(1,2)}$:

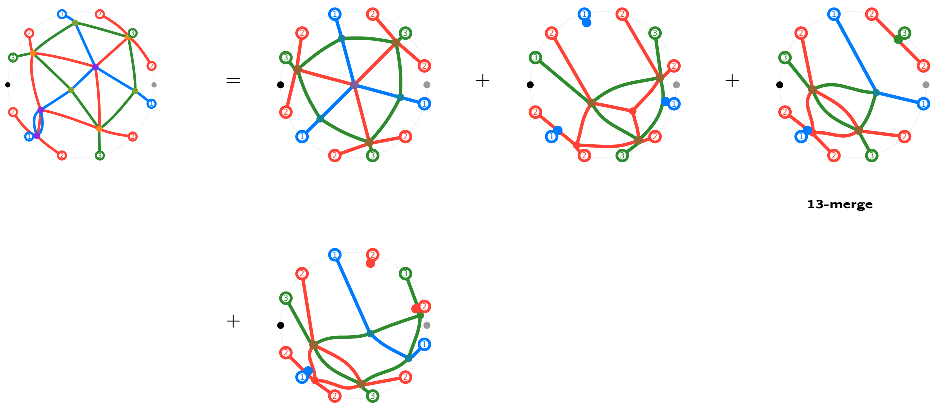


13-dot

RPTU

The $\mathrm{Zamo}(2,9)$ variant

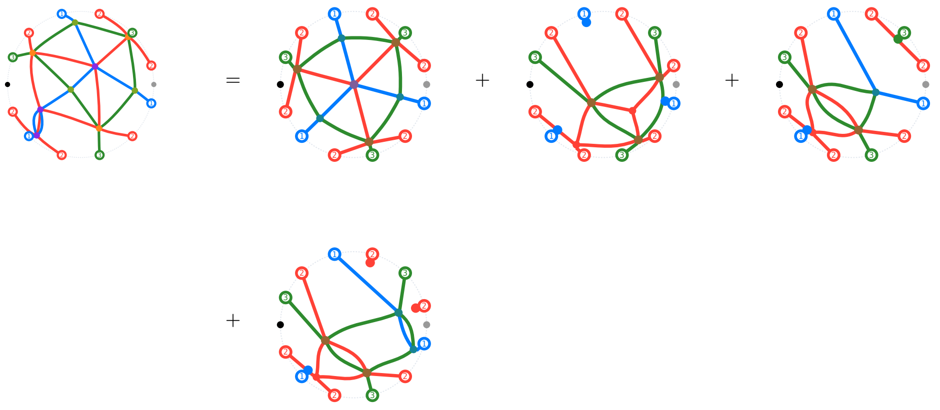
$\mathrm{Zamo}(8,9) \circ \mathrm{Zamo}(1,8) \circ \overline{\mathrm{Zamo}(1,2)}$:



RPTU

The $\mathrm{Zamo}(2,9)$ variant

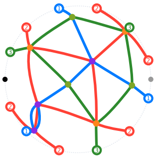
$\mathrm{Zamo}(8,9) \circ \mathrm{Zamo}(1,8) \circ \overline{\mathrm{Zamo}(1,2)}$:



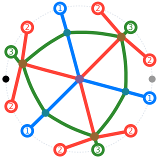
3-merge

The $\mathrm{Zamo}(2, 9)$ variant

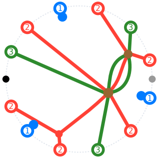
$\mathrm{Zamo}(8, 9) \circ \mathrm{Zamo}(1, 8) \circ \overline{\mathrm{Zamo}(1, 2)}$:



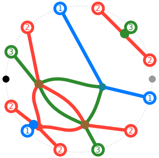
$=$



$+$

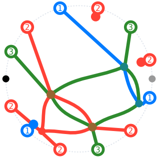


$+$



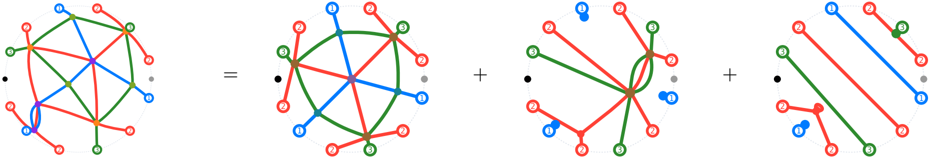
merge into the general node

$+$

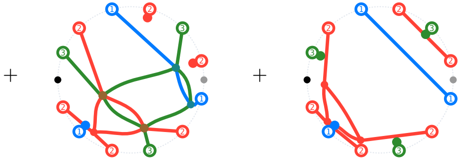


The $\mathrm{Zamo}(2,9)$ variant

$\mathrm{Zamo}(8,9) \circ \mathrm{Zamo}(1,8) \circ \overline{\mathrm{Zamo}(1,2)}$:

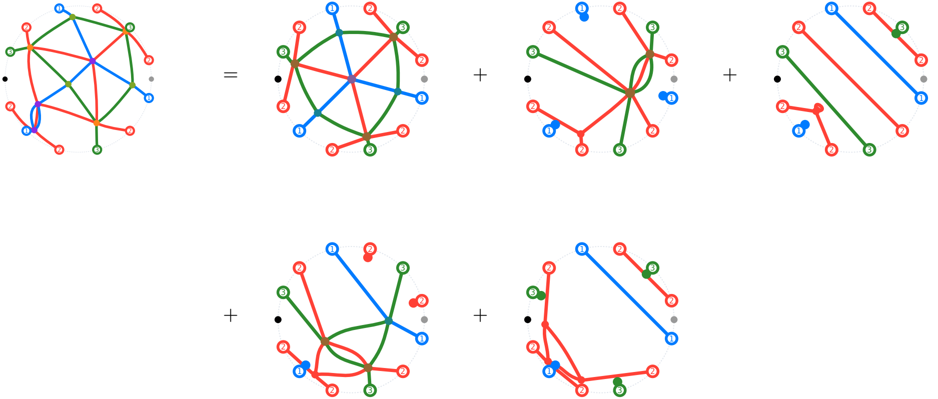


Elias–Jones–Wenzl



The $\mathrm{Zamo}(2,9)$ variant

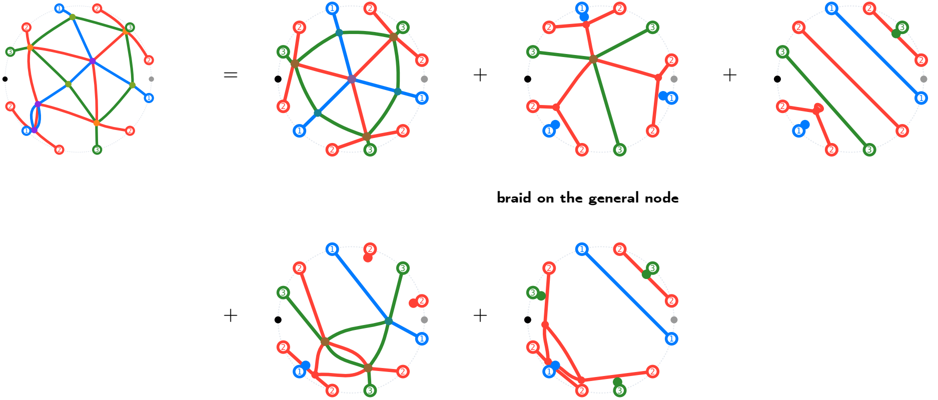
$\mathrm{Zamo}(8,9) \circ \mathrm{Zamo}(1,8) \circ \overline{\mathrm{Zamo}(1,2)}$:



13-merge

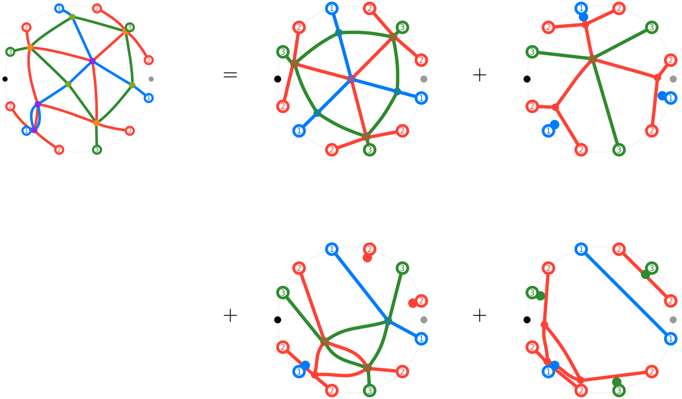
The $\mathrm{Zamo}(2,9)$ variant

$\mathrm{Zamo}(8,9) \circ \mathrm{Zamo}(1,8) \circ \overline{\mathrm{Zamo}(1,2)}$:



The $\mathrm{Zamo}(2,9)$ variant

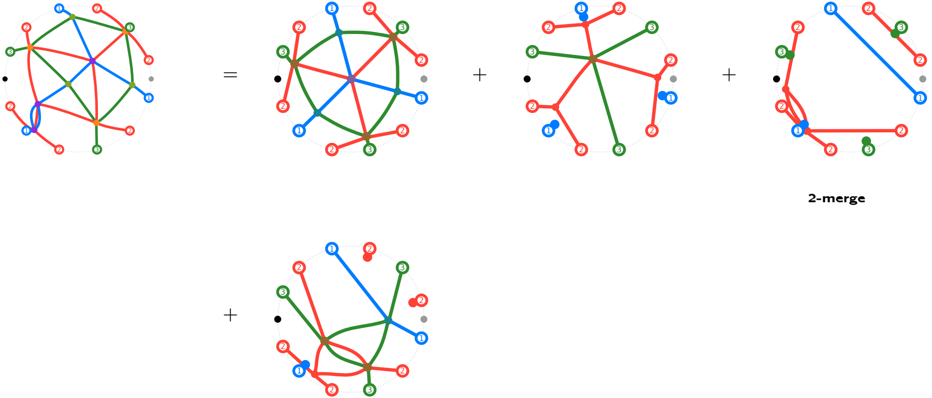
$\mathrm{Zamo}(8,9) \circ \mathrm{Zamo}(1,8) \circ \overline{\mathrm{Zamo}(1,2)}$:



needle $\mapsto$ 0

The $\mathrm{Zamo}(2, 9)$ variant

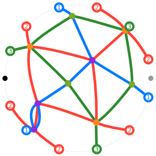
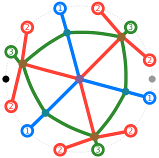
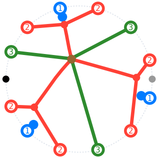
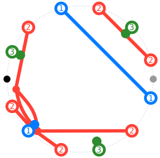
$\mathrm{Zamo}(8, 9) \circ \mathrm{Zamo}(1, 8) \circ \overline{\mathrm{Zamo}(1, 2)}$:

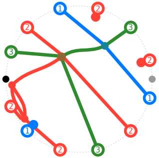


RPTU

The $\mathrm{Zamo}(2,9)$ variant

$\mathrm{Zamo}(8,9) \circ \mathrm{Zamo}(1,8) \circ \overline{\mathrm{Zamo}(1,2)}$:

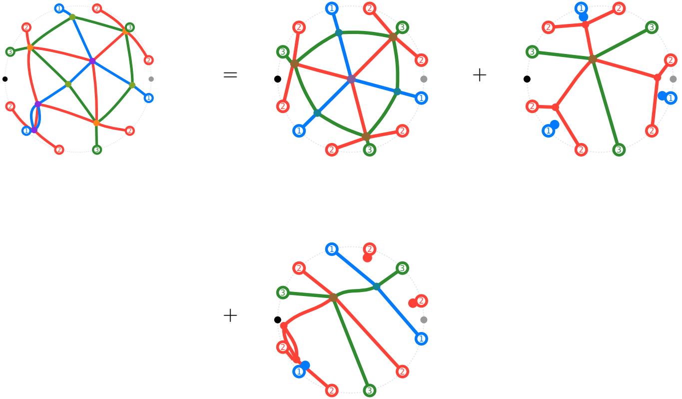
 $=$  $+$  $+$ 

$+$ 

braid-associativity

The $\mathrm{Zamo}(2, 9)$ variant

$\mathrm{Zamo}(8, 9) \circ \mathrm{Zamo}(1, 8) \circ \overline{\mathrm{Zamo}(1, 2)}$:

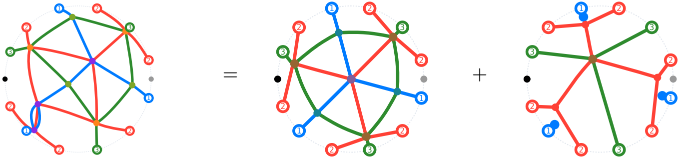


double connection

RPTU

The $\mathrm{Zamo}(2,9)$ variant

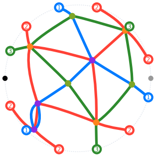
$\mathrm{Zamo}(8,9) \circ \mathrm{Zamo}(1,8) \circ \overline{\mathrm{Zamo}(1,2)}$:



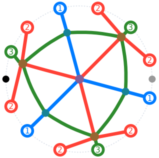
double connection

The $\mathrm{Zamo}(2,9)$ variant

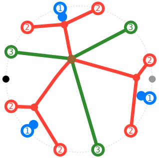
$\mathrm{Zamo}(8,9) \circ \mathrm{Zamo}(1,8) \circ \overline{\mathrm{Zamo}(1,2)}$:



$=$



$+$

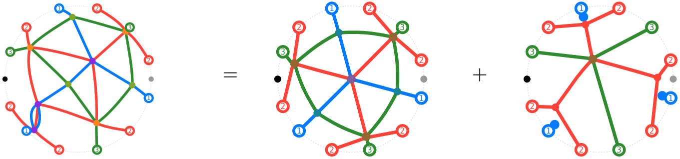


$$\mathrm{Zamo}(\mathbf{2},\mathbf{9}) + \overset{=}{\mathbb{CL}(\mathbf{212}\ldots\mathbf{123},\ \mathbf{100110010011})}$$

RPTU

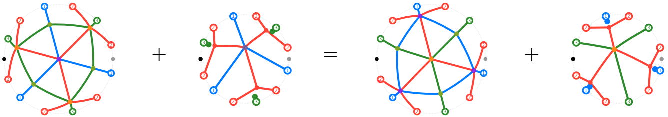
The $\mathrm{Zamo}(2,9)$ variant

$\mathrm{Zamo}(8,9) \circ \mathrm{Zamo}(1,8) \circ \overline{\mathrm{Zamo}(1,2)}$:



$$= \mathrm{Zamo}(\mathbf{2},\mathbf{9}) + \mathbb{CL}(\mathbf{212}\ldots\mathbf{123},\ \mathbf{100110010011})$$

The original Zamolodchikov relation $\mathrm{Zamo}(1,8) = \overline{\mathrm{Zamo}(8,1)}$ is equivalent to:

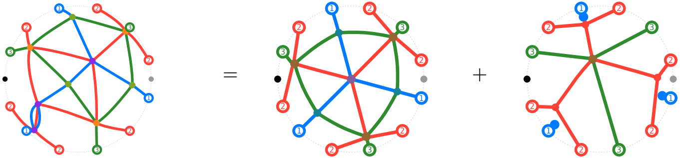


$$\mathrm{Zamo}(2,9) + \mathbb{CL}(212321232123,\ 111000001110) = \overline{\mathrm{Zamo}(9,2)} + \mathbb{CL}(212321232123,\ 100110010011)$$

RPTU

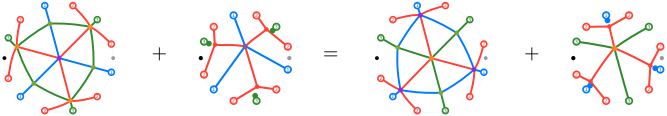
The $\mathrm{Zamo}(2,9)$ variant

$\mathrm{Zamo}(8,9) \circ \mathrm{Zamo}(1,8) \circ \overline{\mathrm{Zamo}(1,2)}$:



$$= \mathrm{Zamo}(\mathbf{2},\mathbf{9}) + \mathbb{CL}(\mathbf{212}\ldots\mathbf{123},\ \mathbf{100110010011})$$

The original Zamolodchikov relation $\mathrm{Zamo}(1,8) = \overline{\mathrm{Zamo}(8,1)}$ is equivalent to:

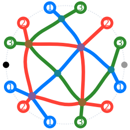


$$\mathrm{Zamo}(2,9) + \mathbb{CL}(212321232123,\ 111000001110) = \overline{\mathrm{Zamo}(9,2)} + \mathbb{CL}(212321232123,\ 100110010011)$$

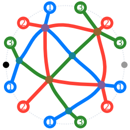
One can extend this to Zamolodchikov regions of size ≥ 7, i.e. tilings with the 123-triangle plus certain conditions.

Zamolodchikov relation in type A_3/H_3

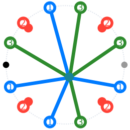
1. The ansatz. At the pair $(121321,\ 321323)$ there are exactly two circular leaves of degree ≤ 0.



$- \lambda_1$



$= \lambda_2$

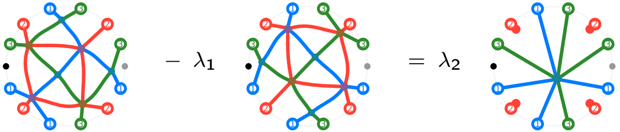


$$\mathrm{Zamo}(\mathbf{1},\mathbf{8}) - \lambda_{\mathbf{1}}\,\overline{\mathrm{Zamo}(\mathbf{8},\mathbf{1})} = \lambda_{\mathbf{2}}\,\mathbb{CL}(\mathbf{121321\ 323123},\ \mathbf{100100\ 000101})$$

RPTU

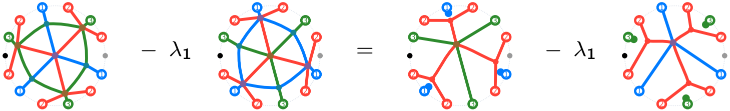
Zamolodchikov relation in type A_3/H_3

1. **The ansatz.** At the pair (121321, 321323) there are exactly two circular leaves of degree ≤ 0.



$\mathrm{Zamo}(\mathbf{1},\mathbf{8}) - \lambda_{\mathbf{1}}\overline{\mathrm{Zamo}(\mathbf{8},\mathbf{1})} = \lambda_{\mathbf{2}}\,\mathbb{CL}(\mathbf{121321\ 323123},\ \mathbf{100100\ 000101})$

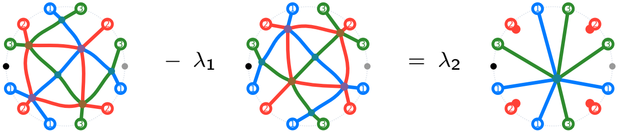
2. **Step 1.** Glue $\mathrm{Zamo}(8,9)$ to the top and $\overline{\mathrm{Zamo}(1,2)}$ to the bottom to get an equivalent equation.



$\mathbb{CL}(\mathbf{212321\ 232123},\ \mathbf{100110\ 010011})$ **and** $\mathbb{CL}(\mathbf{212321\ 232123},\ \mathbf{111000\ 001110})$

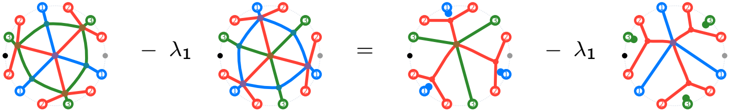
Zamolodchikov relation in type A_3/H_3

1. **The ansatz.** At the pair $(121321, 321323)$ there are exactly two circular leaves of degree ≤ 0.



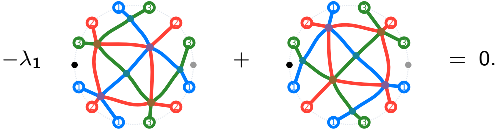
$\mathrm{Zamo}(\mathbf{1},\mathbf{8}) - \lambda_{\mathbf{1}} \overline{\mathrm{Zamo}(\mathbf{8},\mathbf{1})} = \lambda_{\mathbf{2}}\, \mathbb{CL}(\mathbf{121321\,323123},\ \mathbf{100100\,000101})$

2. **Step 1.** Glue $\mathrm{Zamo}(8,9)$ to the top and $\overline{\mathrm{Zamo}(1,2)}$ to the bottom to get an equivalent equation.



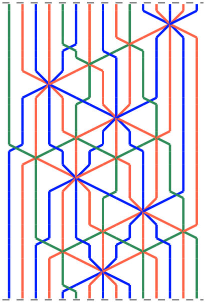
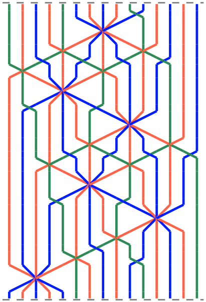
$\mathbb{CL}(\mathbf{212321\,232123},\ \mathbf{100110\,010011})$ **and** $\mathbb{CL}(\mathbf{212321\,232123},\ \mathbf{111000\,001110})$

3. **Seven steps.** Continue for seven steps to get the following.



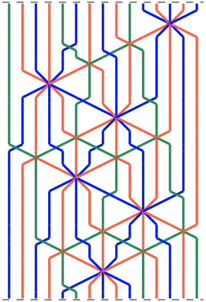
RPTU

Zamolodchikov relation in type A_3/H_3

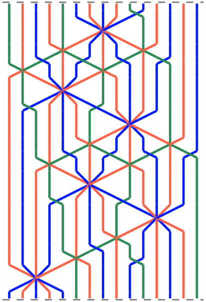
$\mathrm{Zamo}(\mathbf{1}, \mathbf{32}) =$  $\neq$  $= \overline{\mathrm{Zamo}(\mathbf{32}, \mathbf{1})}$

Zamolodchikov relation in type A_3/H_3

$\mathrm{Zamo}(\mathbf{1}, \mathbf{32}) =$



$\neq$



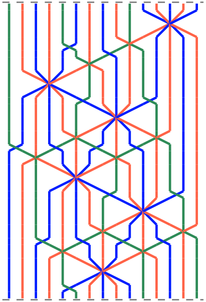
$= \overline{\mathrm{Zamo}(\mathbf{32}, \mathbf{1})}$

And in H_3? We hope to achieve the same. Here $\mathrm{Zamo}(1, 32) - \lambda_{\mathbf{1}} \overline{\mathrm{Zamo}(32, 1)}$ is a linear system in many diagrams: for $x = 121213212132123$ and $y = x^{-\mathbf{1}} = 321231212312121$ we have

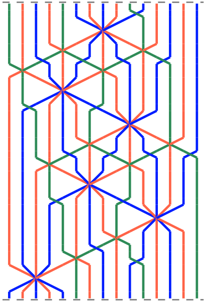
$$\mathrm{rk}\,\mathrm{Hom}_{\mathcal{H}_{BS}}(x, y) = v^{-\mathbf{8}} + 34\, v^{-\mathbf{6}} + 578\, v^{-\mathbf{4}} + 6644\, v^{-\mathbf{2}} + 71160\, v^{\mathbf{0}} + \dots .$$

Zamolodchikov relation in type A_3/H_3

$\mathrm{Zamo}(\mathbf{1},\mathbf{32}) =$



$\neq$



$= \overline{\mathrm{Zamo}(\mathbf{32},\mathbf{1})}$

And in H_3? We hope to achieve the same. Here $\mathrm{Zamo}(1,32) - \lambda_{\mathbf{1}}\overline{\mathrm{Zamo}(32,1)}$ is a linear system in many diagrams: for $x = 121213212132123$ and $y = x^{-1} = 321231212312121$ we have

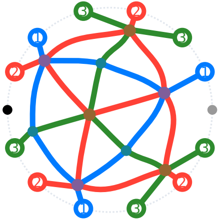
$$\operatorname{rk}\mathrm{Hom}_{\mathcal{H}_{BS}}(x,y) = v^{-8} + 34\,v^{-6} + 578\,v^{-4} + 6644\,v^{-2} + 71160\,v^{0} + \dots .$$

The software package and a paper on circular leaves should appear soon; the sooner, the faster somebody might offer me a nice postdoc position (October 2026 seems to be a good month to start something new).

https://github.com/rogeliam/DiagrammaticHecke.jl

RPTU

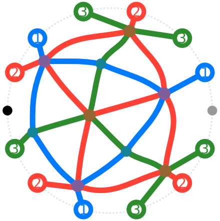
Big example from the title slide



$=$

RPTU

Big example from the title slide



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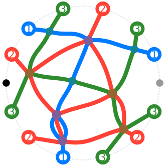
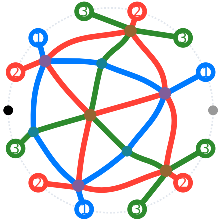


1. the Zamolodchikov relation

RPTU

Big example from the title slide



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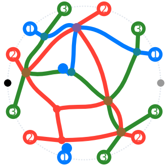

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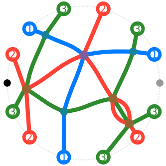
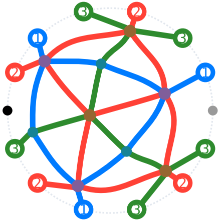


2. Elias–Jones–Wenzl

Big example from the title slide



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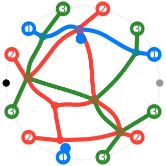

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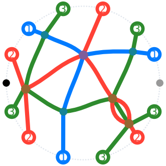
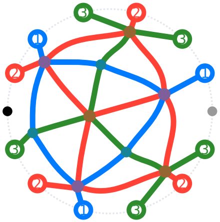


3. 13-dot

Big example from the title slide



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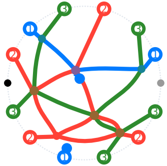

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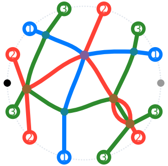
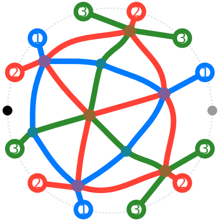


4. 2-merge

RPTU

Big example from the title slide



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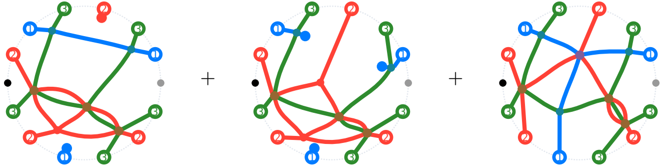
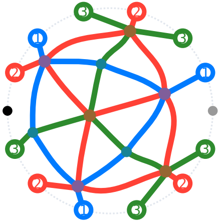


5. 12-dot into braid

RPTU

Big example from the title slide



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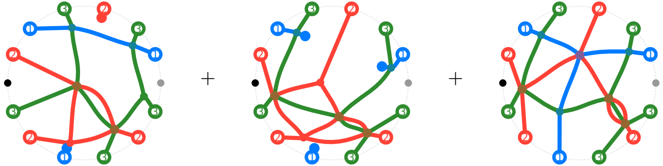
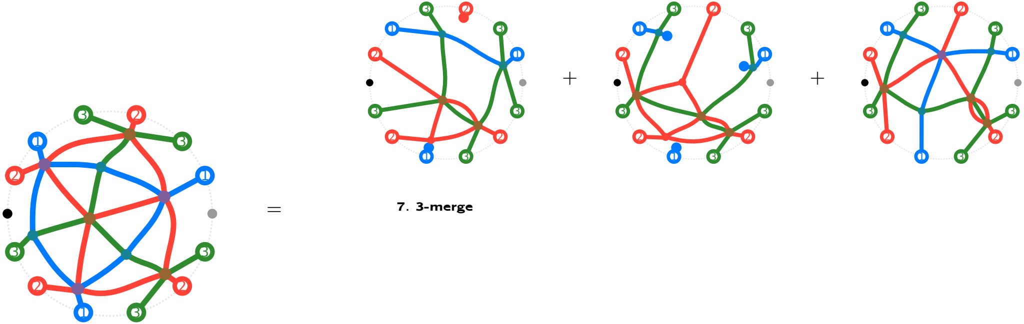


6. braid-associativity

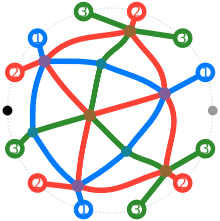
RPTU

Big example from the title slide



7. 3-merge

Big example from the title slide



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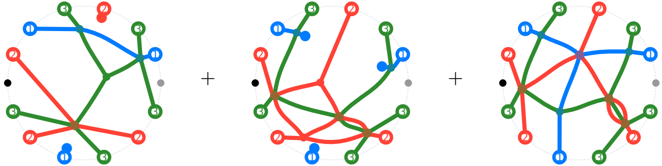
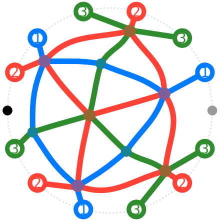


8. braid-associativity

RPTU

Big example from the title slide



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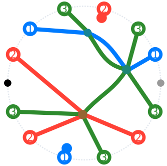

9. 3-merge

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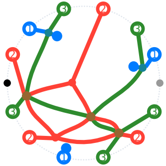

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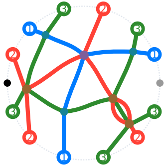
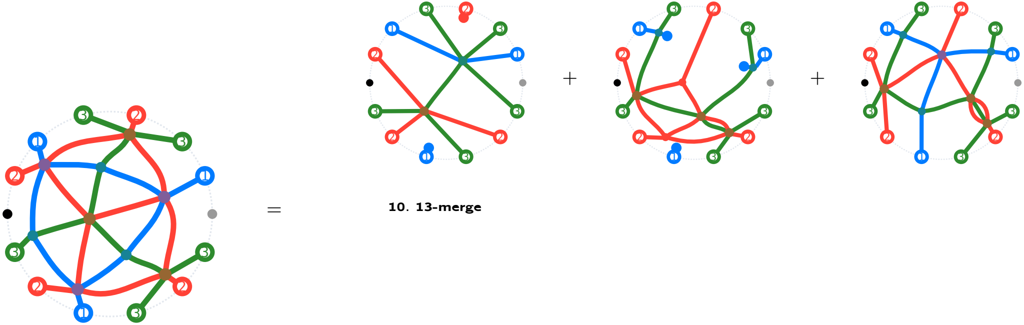
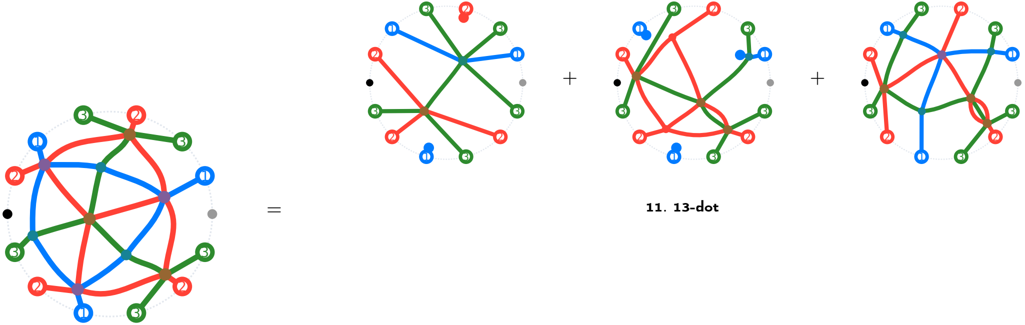


Big example from the title slide



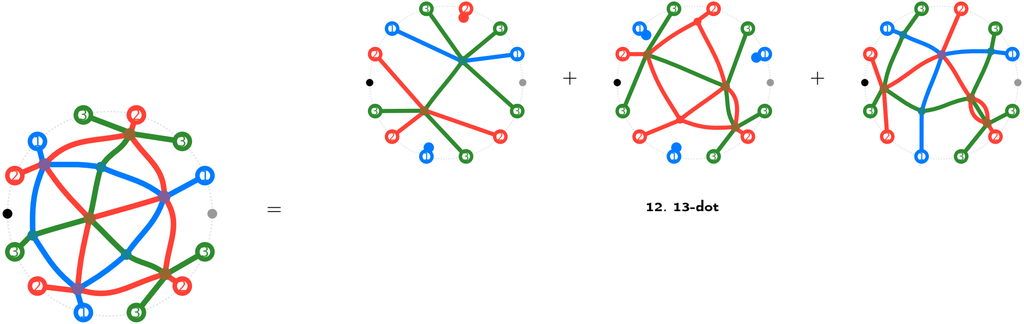
RPTU

Big example from the title slide



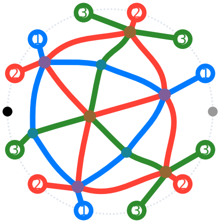
11. 13-dot

Big example from the title slide



12. 13-dot

Big example from the title slide



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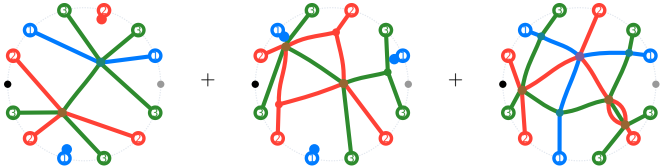
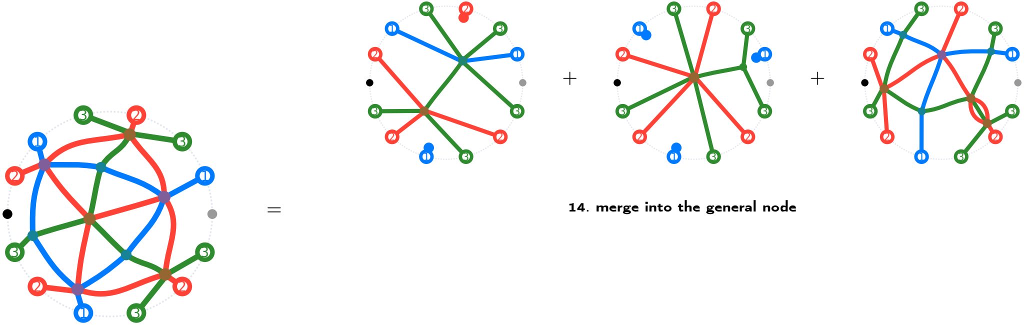


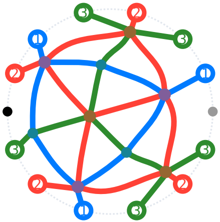
13. braid-associativity

Big example from the title slide



14. merge into the general node

Big example from the title slide



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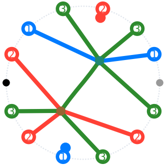

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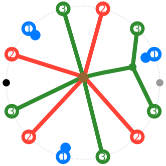

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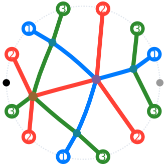

15. Elias–Jones–Wenzl

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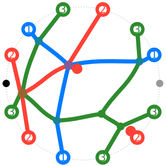
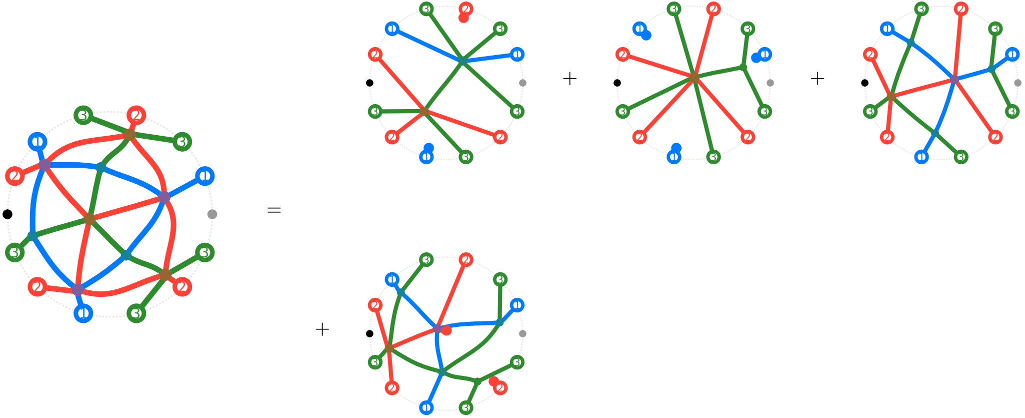
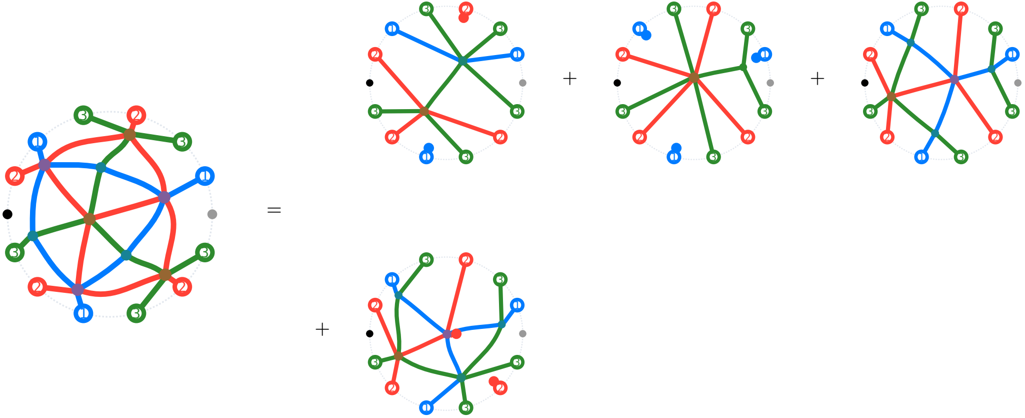


Big example from the title slide



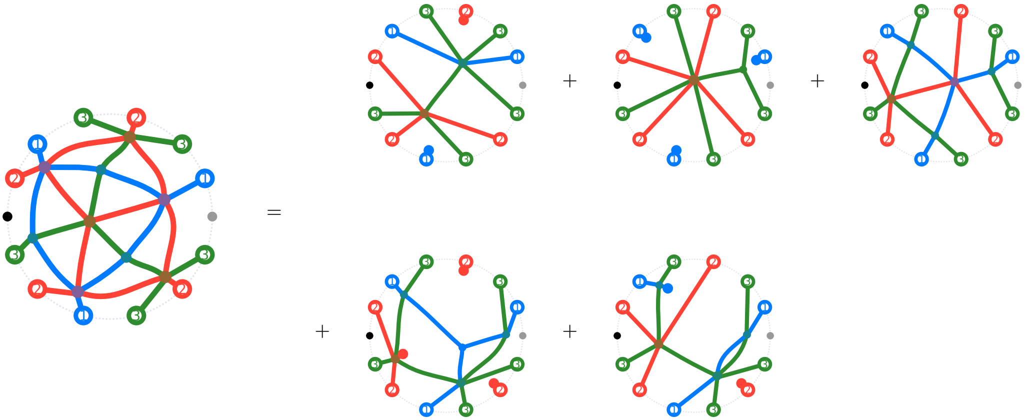
16. 3-merge

Big example from the title slide



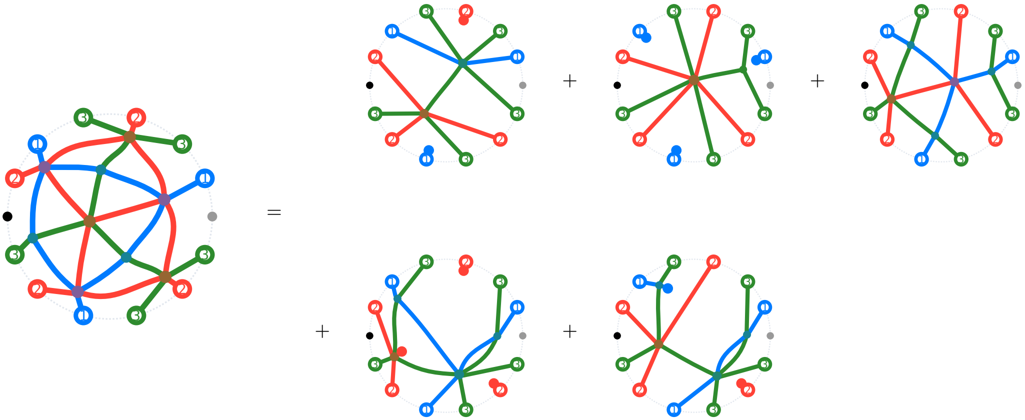
17. 3-merge

Big example from the title slide



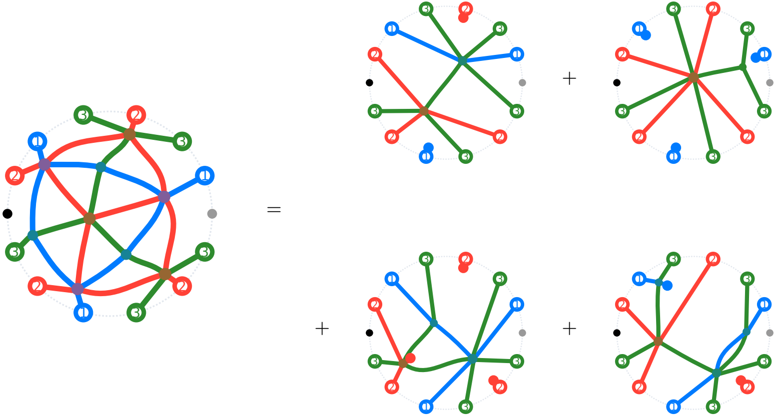
18. 12-dot into braid

Big example from the title slide



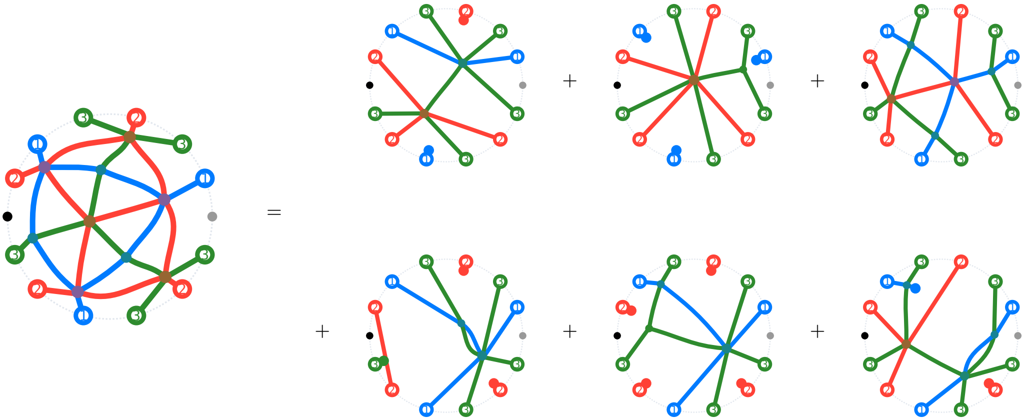
19. 1-merge

Big example from the title slide



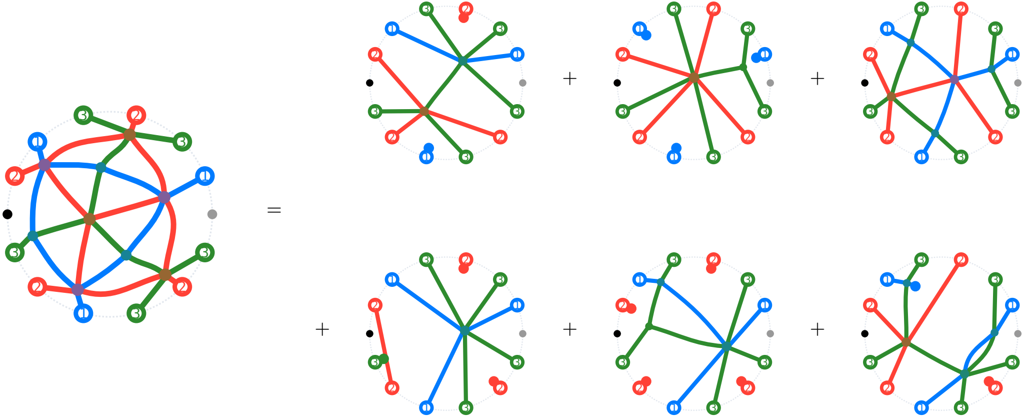
20. 13-merge

Big example from the title slide



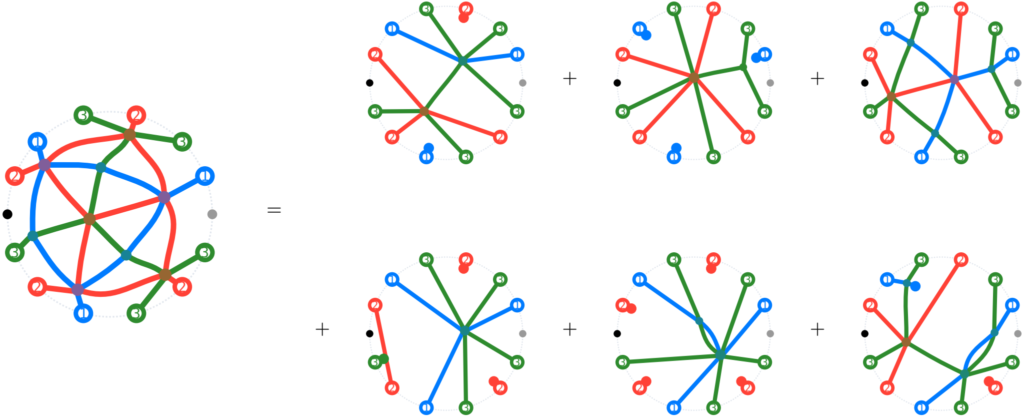
21. 23-dot into braid

Big example from the title slide



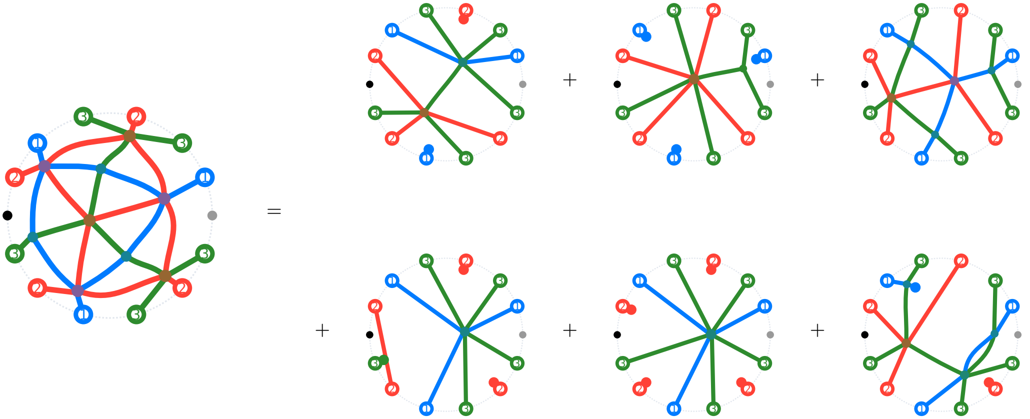
22. 13-merge

Big example from the title slide



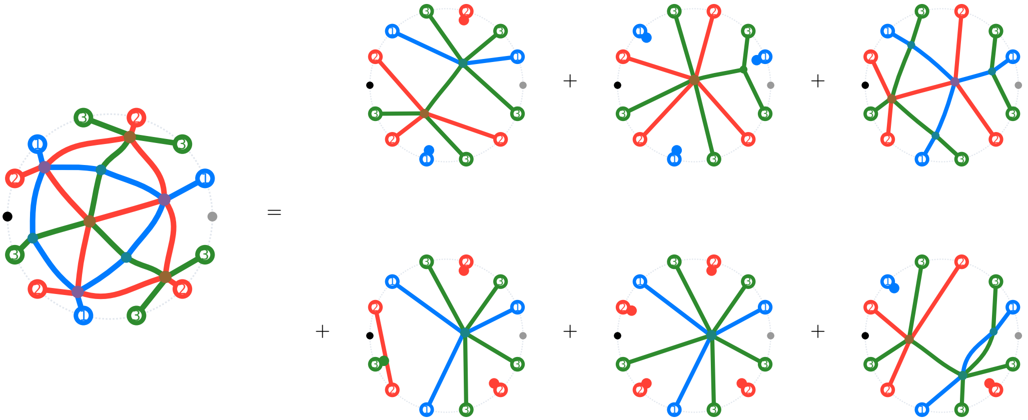
23. 3-merge

Big example from the title slide



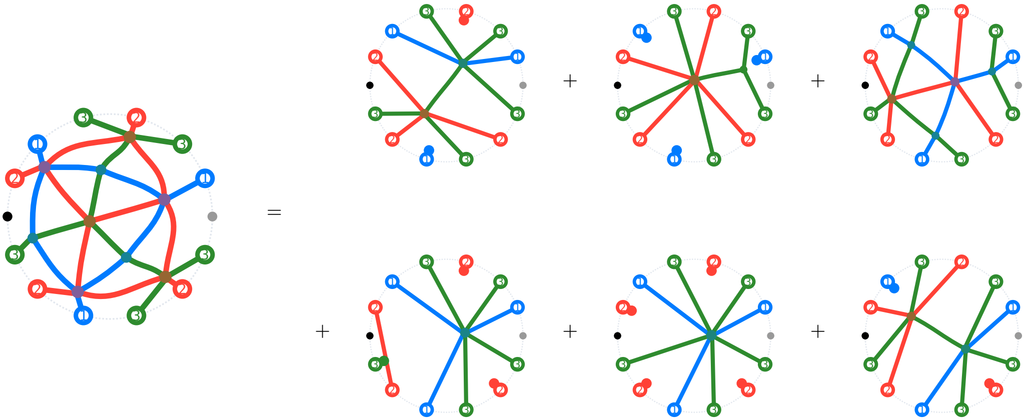
24. 13-merge

Big example from the title slide



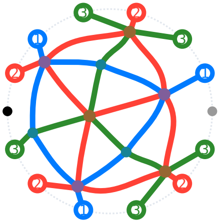
25. 13-dot

Big example from the title slide



26. 13-merge

Big example from the title slide



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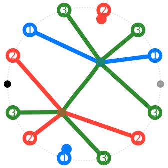

$\mathbb{CL}(\mathbf{110110\ 100111})$

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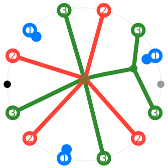

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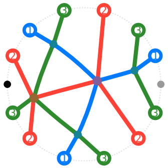

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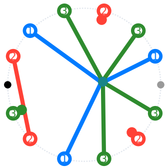

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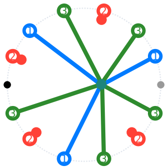

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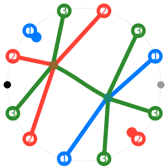

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